A SHARP ERROR BOUND IN TERMS OF AN AVERAGED MODULUS OF SMOOTHNESS FOR FOURIER LAGRANGE COEFFICIENTS

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Abstract: This paper discusses an error bound for the approximation of Fourier coefficients by Fourier Lagrange coefficients in terms of an averaged modulus of smoothness. The sharpness of this estimate is shown as an application of a quantitative resonance principle by utilizing the aliasing phenomenon that occurs in the context of discrete Fourier transformation. The scenario is used to compare the averaged modulus with classical uniform and integral moduli of smoothness.

Key words: *Averaged modulus of smoothness, τ-modulus, sharp error bounds, resonance principle, aliasing*

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1 Introduction

We estimate the remainder between Fourier coefficients of a 2π periodic function f and their discrete counterparts computed by a discrete Fourier transform.

To this end, $B_{2\pi}$ denotes the space of 2π periodic bounded and $R_{2\pi}$ denotes the space of 2π periodic Riemann integrable functions. Furthermore, let $C_{2\pi}$ be the space of continuous 2π periodic functions, i.e. $C_{2\pi} \subset R_{2\pi} \subset B_{2\pi}$.

For $f \in R_{2\pi}$ the typical discretization $(f^\wedge)_n^*(k)$ of Fourier coefficients

$$f^\wedge(k) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\, dt$$

is given by a quadrature formula on $2n+1$ equidistant knots $u_{j,n} := j \cdot \frac{2\pi}{2n+1}$, $-n \le k \le n$:

$$(f^\wedge)_n^*(k) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(u_{j,n}) e^{-iku_{j,n}}.$$

$\sum_{k=-n}^{n} (f^\wedge)_n^*(k) e^{jkt}$ is the Lagrange interpolation polynomial of degree at most n that interpolates f at the knots $u_{j,n}$.

The error rate of $|f^\wedge(k) - (f^\wedge)_n^*(k)|$ depends on the smoothness of f that is measured in terms of moduli of smoothness. This is a fundamental concept of Approximation Theory that discusses first and higher differences that – in contrast to derivatives – always exist (cf. [15], and [1] for computation of moduli).

Let $f \in B_{2\pi}$ and $r \in \mathbb{N}$ be a natural number. The r-th difference of f at point t on the real axis $\mathbb{R}$ is defined as

$$\Delta_h^1 f(t) := f(t+h) - f(t), \quad \Delta_h^r f(t) := \Delta_h^1 \Delta_h^{r-1} f(t),\ r > 1, \text{ or}$$

$$\Delta_h^r f(t) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(t+jh).$$

The r-th uniform modulus of smoothness (or modulus of continuity) is the smallest upper bound of the absolute value of r-th differences:

$$\omega_r(\delta, f) := \sup \left\{ |\Delta_h^r f(t)| : t \in [0, 2\pi],\ 0 < h \leq \delta \right\}.$$

This modulus typically is used for (uniformly) continuous functions for which it converges to zero with a certain rate when $\delta \to 0+$.

Since we discuss Fourier coefficients that are defined via an integral we need to introduce the integral modulus

$$\omega_r(\delta, f, L_{2\pi}^1) := \sup_{0<h\leq\delta} \int_0^{2\pi} |\Delta_h^r f(t)|\, dt,$$

defined for Lebesgue integrable, i.e. especially for Riemann integrable, 2π periodic functions f. Instead of the supremum norm here the $L_{2\pi}^1$ norm $\|f\|_{L_{2\pi}^1} := \int_0^{2\pi} |f(t)|\, dt$ is used, where $L_{2\pi}^1$ is the space of 2π periodic measurable functions having finite $L_{2\pi}^1$ norm. Obviously,

$$\omega_r(\delta, f, L_{2\pi}^1) \leq 2\pi \omega_r(\delta, f).$$

With the help of the integral modulus we now derive a naive error bound for Fourier Lagrange coefficients.

Lemma 1.1 *Let $f \in R_{2\pi}$ be a function that can be represented by its Fourier series on the set $\{u_{j,n} : n \in \mathbb{N},\ j \in \{0, 1, 2, \ldots, 2n\}\}$. Additionally, for $r \in \mathbb{N}$ and $\alpha > 1$ we assume that $\omega_r(\delta, f, L_{2\pi}^1) = \mathcal{O}(\delta^\alpha)$ $(\delta \to 0+)$. Then there exists a constant $C_r > 0$ only dependent on r (and not on k or n) so that for $n \in \mathbb{N}$ and $-n \leq k \leq n$*

$$|f^\wedge(k) - (f^\wedge)_n^*(k)| = \left| \sum_{m\in\mathbb{Z}\setminus\{0\}} f^\wedge(k + m(2n+1)) \right| \leq C_r \frac{1}{n^\alpha}. \tag{1.1}$$

For the sake of completeness we give a short proof of this estimate that is based on the Riemann Lebesgue type estimate with orders (cf. [2, p.168], $k \neq 0$)

$$|f^\wedge(k)| \leq C_r \omega_r \left(\frac{1}{|k|}, f, L_{2\pi}^1 \right), \tag{1.2}$$

where the constant $0 < C_r < \infty$ only depends on $r \in \mathbb{N}$. Because of (1.2) and $\alpha > 1$ the Fourier series is absolutely convergent:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left| f^\wedge(k) e^{ikt} \right| &\leq |f^\wedge(0)| + 2C_r \sum_{k=1}^{\infty} \omega_r \left(\frac{1}{k}, f, L_{2\pi}^1 \right) \\ &\leq |f^\wedge(0)| + 2C_r \sum_{k=1}^{\infty} C k^{-\alpha} < \infty. \end{aligned}$$

At a node $t = u_{j,n}$ all functions $e^{i[k+m(2n+1)]t}$ have the same value $e^{iku_{j,n}}$ and as assumed $f(u_{j,n})$ is the limit of the Fourier series at the point $u_{j,n}$. Since the series is absolutely convergent we can change order of summation and find for each sample point $u_{j,n}$:

$$\begin{aligned} f(u_{j,n}) &= \sum_{k=-\infty}^{\infty} f^\wedge(k) e^{iku_{j,n}} = \sum_{k=-n}^{n} \sum_{m=-\infty}^{\infty} f^\wedge(k+m(2n+1)) e^{i[k+m(2n+1)]u_{j,n}} \\ &= \sum_{k=-n}^{n} \left[\sum_{m=-\infty}^{\infty} f^\wedge(k+m(2n+1)) \right] e^{iku_{j,n}}. \end{aligned}$$

Since the interpolation polynomial $\sum_{k=-n}^{n} (f^\wedge)_n^*(k) e^{ikt}$ of degree at most n for the knots $u_{j,n}$ is unique, the following representation of Fourier Lagrange coefficients holds true:

$$(f^\wedge)_n^*(k) = \sum_{m=-\infty}^{\infty} f^\wedge(k+m(2n+1)). \tag{1.3}$$

The distance between (1.3) and $f^\wedge(k)$ is the well known aliasing error. Again, we apply Riemann Lebesgue type estimate with orders (1.2) for $-n \le k \le n$:

$$\begin{aligned} |f^\wedge(k) - (f^\wedge)_n^*(k)| &\le \sum_{m=1}^{\infty} \left[|f^\wedge(k+m(2n+1))| + |f^\wedge(k-m(2n+1))| \right] \\ &\le C_r \sum_{m=1}^{\infty} \left[\omega_r \left(\frac{1}{k+m(2n+1)}, f, L^1_{2\pi} \right) + \omega_r \left(\frac{1}{m(2n+1)-k}, f, L^1_{2\pi} \right) \right] \\ &\le 2C_r \sum_{m=1}^{\infty} \omega_r \left(\frac{1}{nm}, f, L^1_{2\pi} \right). \end{aligned}$$

Because of $\omega_r(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha)$ with $\alpha > 1$ one gets

$$|f^\wedge(k) - (f^\wedge)_n^*(k)| \le C_1 \frac{1}{n^\alpha} \sum_{m=1}^{\infty} \frac{1}{m^\alpha} = C_2 \frac{1}{n^\alpha}.$$

■

Please note that, if $f \in C_{2\pi}$ and $\omega_r(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha)$, the Fourier series is uniformly convergent to f thus fulfilling the requirements of Lemma 1.1.

It seems to be natural that some additional smoothness like $\alpha > 1$ is needed in connection with pointwise interpolation. But what can we obtain for $\alpha \le 1$? For example $\alpha = 1$ occurs for the the non-continuous but piecewise constant function

$$f_0(t) := \begin{cases} 0, & t = -\pi \text{ und } t = 0, \\ 1, & t \in]-\pi, 0[, \\ -1, & t \in]0, \pi[. \end{cases} \tag{1.4}$$

that can be written as the convergent series $f_0(t) = -\sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin((2k-1)t)$.

The next sections deal with the averaged modulus and its properties. Then we derive an error estimate for Fourier Lagrange coefficients that does not require extra assumptions on the smoothness of f. The last section shows that the error estimate can not be improved. We establish this sharpness in terms of counterexamples via a gliding hump theorem that uses a resonance condition. To show resonance we utilise the aliasing effect of the discrete Fourier transform.

2 Delta norm and averaged modulus of smoothness

Point functionals like interpolation operators are not bounded with respect to the $L^1_{2\pi}$-norm. Therefore one can not write related error bounds in terms of $\omega_r\left(\delta, f, L^1_{2\pi}\right)$. The averaged modulus of smoothness or τ-modulus is better suited. For certain classes of functions it behaves similar to $\omega_r\left(\delta, f, L^1_{2\pi}\right)$ but shows significant better rates than the uniform modulus. Instead of dealing with a sup-norm we follow [3] and [16] and use the δ-norm

$$\|f\|_\delta := \int_0^{2\pi} M(\delta, f, t)\, dt,$$

where a local supremum is defined as (cf. [13])

$$M(\delta, f, x) := \|f\|_{B[x-\delta, x+\delta]} := \sup\{|f(t)| : t \in [x-\delta, x+\delta]\}.$$

Because of [8] we are allowed to define the δ-norm with a Riemann integral instead of an upper Riemann or Lebesgue integral. For $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, $\lambda > 0$, one has (cf. [13]):

$$\|f\|_{m\delta} \leq m\|f\|_\delta, \quad \|f\|_{\lambda\delta} \leq (1+\lambda)\|f\|_\delta. \tag{2.1}$$

Single function values do influence the δ-norm which usually is used to measure errors in connection with approximation processes for Riemann integrable functions. In this function space smoothness is measured by the τ-modulus. To introduce this modulus we deal with the r-th local modulus of smoothness of f at a point x. For $\delta > 0$ it is defined as

$$\omega_r(\delta, f, x) := \sup\left\{|\Delta_h^r f(t)| : t, t+rh \in \left[x - r\frac{\delta}{2}, x + r\frac{\delta}{2}\right], 0 < h \leq \delta\right\}.$$

If f is measurable then the local modulus is measurable as well (see [15, p.13]). Furthermore, in [8] it is shown that the r-th local modulus of smoothness of a Riemann integrable function on a compact interval $[a, b]$ is Riemann integrable itself. By selecting $[a, b] := \left[-r\frac{\delta}{2}, 2\pi + r\frac{\delta}{2}\right]$ this result holds true for $\omega_r(\delta, f, \cdot)$ as defined here. Therefore, apart from Sendov's and Korovkin's definition of τ-moduli via the Lebesgue integral (cf. [15, pp.12]), for $f \in R_{2\pi}$ we can use the Riemann integral to define (cf. [3], [7], [11], [16]):

$$\tau_r(\delta, f) := \int_0^{2\pi} \omega_r(\delta, f, t)\, dt.$$

The local modulus behaves similar but not exactly like the corresponding uniform modulus because the interval of the local modulus depends on r and δ (see [15, p.8]):

$$\tau_r(n\delta, f) \leq (2n)^{r+1}\tau_r(\delta, f), \tag{2.2}$$

$$\tau_r(\delta, f) \leq 2\tau_{r-1}\left(\frac{r}{r-1}\delta, f\right) \leq 2\tau_{r-1}(2\delta, f) \leq 2^{2r+1}\tau_{r-1}(\delta, f), \ r > 1. \tag{2.3}$$

For differentiable functions differences can be replaced by derivatives. Let $C^{(r)}_{2\pi}$ the space of 2π periodic r-times continuously differentiable functions with norm $\|f\|^{(r)}_{2\pi} :=$

$\sum_{k=0}^{r} \|f^{(k)}\|_{B_{2\pi}}$, where $\|f^{(k)}\|_{B_{2\pi}} = \sup_{t\in[0,2\pi]} |f^{(k)}(t)|$. For $f \in C_{2\pi}^{(1)}$ and $r > 1$ there holds true (see [4], cf. [15, p.2])

$$\omega_r(\delta, f) \leq \delta\omega_{r-1}(\delta, f'), \tag{2.4}$$

$$\omega_{r-1}(\delta, f') \leq C_r \int_0^\delta \omega_r(t, f)\frac{1}{t^2}\, dt. \tag{2.5}$$

For the integral modulus and averaged modulus (2.4) also is true (cf. [15, p.8]): If $f \in L_{2\pi}^1$ is n-times absolutely continuous with n-th derivative in $L_{2\pi}^1$, $1 \leq n < r$, it is well known that

$$\omega_r(\delta, f, L_{2\pi}^1) \leq \delta^n\omega_{r-n}(\delta, f^{(n)}, L_{2\pi}^1), \tag{2.6}$$

$$\tau_r(\delta, f) \leq C_{r,n}\delta^n\tau_{r-n}(\delta, f^{(n)}). \tag{2.7}$$

As a consequence of Marchaud's inequality (for the averaged modulus cf. [15, p.12]) there is a relationship between the rates of moduli defined for higher and lower differences. For $f \in R_{2\pi}$ and $r \in \mathbb{N}$:

$$\tau_{r+1}(\delta, f) = \mathcal{O}(\delta^\alpha) \Longrightarrow \tau_r(\delta, f) = \begin{cases} \mathcal{O}(\delta^\alpha), & \alpha < r, \\ \mathcal{O}(|\ln\delta|\delta^r), & \alpha = r, \\ \mathcal{O}(\delta^r), & \alpha > r. \end{cases} \tag{2.8}$$

Estimate (2.3) immediately shows that on the other hand ($r, n \in \mathbb{N}$, $\alpha > 0$)

$$\tau_r(\delta, f) = \mathcal{O}(\delta^\alpha) \Longrightarrow \tau_{r+n}(\delta, f) = \mathcal{O}(\delta^\alpha). \tag{2.9}$$

The same implications hold true for the uniform modulus ($f \in B_{2\pi}$) and the integral modulus ($f \in L_{2\pi}^1$).

Also, all moduli of smoothness show saturation behaviour: If $\tau_r(\delta, f) = o(\delta^r)$ then $\omega_r(\delta, f, L_{2\pi}^1) = o(\delta^r)$, and it is well known that the 2π periodic function f has to be a constant a.e. If a τ-modulus on an interval $[a, b]$ shows saturation behaviour, then the function has to be an algebraic polynomial of degree $r - 1$ without the restriction a.e. The proof given in [7] can be extended to the 2π periodic case without modification. Therefore ($f \in B_{2\pi}$):

$$\tau_r(\delta, f) = o(\delta^r) \Longleftrightarrow f = c \quad \text{for a constant } c.$$

3 Comparison between different moduli

Obviously, there is

$$\omega_r(\delta, f, L_{2\pi}^1) \leq \tau_r(\delta, f) \leq 2\pi\omega_r(\delta, f). \tag{3.1}$$

For discontinuous functions the moduli indeed can show different rates, for example (see (1.4))

$$\omega_r(\delta, f_0, L_{2\pi}^1) = \mathcal{O}(\delta), \quad \tau_r(\delta, f_0) = \mathcal{O}(\delta), \quad \omega_r(\delta, f_0) \neq o(1).$$

Consider $f_1(j2\pi) := 1$ and $f_1(t) := 0$ elsewhere. Then exact values of the moduli directly follow from the definition of r-th differences:

$$\omega_r(\delta, f_1, L^1_{2\pi}) = 0, \quad \tau_r(\delta, f_1) = \binom{r}{\lfloor \frac{r}{2} \rfloor} \cdot r\delta, \quad \omega_r(\delta, f_1) = \binom{r}{\lfloor \frac{r}{2} \rfloor},$$

where $\lfloor \frac{r}{2} \rfloor$ is the biggest integer less or equal $\frac{r}{2}$.

In what follows we compare uniform and averaged moduli. For a given $\varepsilon > 0$ we find some point t_0 where $|\omega_r(\delta, f, t_0) - \omega_r(\delta, f)| < \varepsilon$. Since $\omega_r(2\delta, f, t) \geq \omega_r(\delta, f, t_0)$ for all $t \in \left[t_0 - r\frac{\delta}{2}, t_0 + r\frac{\delta}{2}\right]$, we have

$$\tau_r(2\delta, f) \geq \int_{t_0 - r\frac{\delta}{2}}^{t_0 + r\frac{\delta}{2}} \omega_r(2\delta, f, t)\, dt \geq \int_{t_0 - r\frac{\delta}{2}}^{t_0 + r\frac{\delta}{2}} \omega_r(\delta, f, t_0)\, dt \geq r\delta \left[\omega_r(\delta, f) - \varepsilon\right]$$

so that with (2.2)

$$\omega_r(\delta, f) \leq \frac{1}{r\delta}\tau_r(2\delta, f) \leq \frac{4^{r+1}}{r\delta}\tau_r(\delta, f). \tag{3.2}$$

Therefore, if $\tau_r(\delta, f) = \mathcal{O}(\delta^{1+\alpha})$ for $0 < \alpha < 1$ then $\omega_r(\delta, f) = \mathcal{O}(\delta^\alpha)$. Implication (2.8) also holds true for the uniform modulus: $\omega_1(\delta, f) = \mathcal{O}(\delta^\alpha)$. This in turn means that f is continuous. So if one expects for $\tau_r(\delta, f)$ a higher rate of convergence than $\mathcal{O}(\delta)$, one has to assume continuity.

Because of (3.2) the rate of convergence of $\tau_r(\delta, f)$ in comparison to $\omega_r(\delta, f)$ can not be better than one additional power of δ. But on the other hand this best possible rate can be obtained for a certain class of functions. Indeed, for absolutely continuous, nonconcave functions on a compact interval $[a, b]$ this has been shown in [7] by using second differences.

Let us look at the 2π periodic continuous functions

$$g_\alpha(t) := |\sin t|^\alpha, \quad 0 < \alpha \leq 1.$$

By splitting up the integration interval $I := [0, 2\pi]$ of the τ-modulus into the intervals $I_1 := [0, \delta]$, $I_2 := \left[\delta, \frac{\pi}{2} - \delta\right]$, $I_3 := \left[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right]$, $I_4 := \left[\frac{\pi}{2} + \delta, \pi - \delta\right]$, $I_5 := [\pi - \delta, \pi + \delta]$, $I_6 := \left[\pi + \delta, \frac{3\pi}{2} - \delta\right]$, $I_7 := \left[\frac{3\pi}{2} - \delta, \frac{3\pi}{2} + \delta\right]$, $I_8 := \left[\frac{3\pi}{2} + \delta, 2\pi - \delta\right]$ and $I_9 := [2\pi - \delta, 2\pi]$ one gets

$$\begin{aligned} \tau_2(\delta, g_\alpha) &\leq 8\delta \|\omega_2(\delta, g_\alpha, \cdot)\|_{B_{2\pi}} + \int_{I_2 \cup I_4 \cup I_6 \cup I_8} \omega_2(\delta, g_\alpha, t)\, dt \\ &\leq 8\delta\omega_2(\delta, g_\alpha) + 4\int_{I_1} \omega_2(\delta, g_\alpha, t)\, dt. \end{aligned}$$

On I_1 functions $-g_\alpha$ are nonconcave because $-g'_\alpha(t) = -\alpha \cos t \sin^{\alpha-1} t$ and

$$g''_\alpha(t) = -\alpha\left[-\sin t \sin^{\alpha-1} t + (\alpha - 1)\cos^2 t \sin^{\alpha-2} t\right] > 0.$$

According to [7] we get

$$\int_{I_1} \omega_2(\delta, g_\alpha, t)\, dt = \int_{I_1} \omega_2(\delta, -g_\alpha, t)\, dt \leq 8\delta\omega_1(\delta, -g_\alpha)$$

so that we finally have

$$\begin{aligned}\tau_2(\delta, g_\alpha) &\leq 8\delta\left[\omega_2(\delta, g_\alpha) + 4\omega_1(\delta, g_\alpha)\right] \leq 8\delta\left[2\omega_1(\delta, g_\alpha) + 4\omega_1(\delta, g_\alpha)\right] \\ &= 48\delta\omega_1(\delta, g_\alpha) \leq 48\delta^{1+\alpha}. \end{aligned} \tag{3.3}$$

On the other hand $\omega_2(\delta, g_\alpha)$ shows no better rate than δ^α. For $\delta < \frac{\pi}{2}$ a second difference at the point 0 gives:

$$\omega_2(\delta, g_\alpha) \geq |g_\alpha(-\delta) - 2g_\alpha(0) + g_\alpha(\delta)| = 2\sin^\alpha(\delta) \geq 2\left(\frac{2}{\pi}\right)^\alpha \delta^\alpha. \tag{3.4}$$

Next we compare integral and averaged moduli. We start with a weak type inequality that holds true for continuous functions f (cf. [15, p.18] and the literature cited there)

$$\tau_r(\delta, f) \leq C_r\delta \int_0^\delta t^{-2}\omega_r(t, f, L^1_{2\pi})\,dt.$$

So if continuous f satisfies Lipschitz condition $\omega_r(t, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha)$ for $\alpha > 1$, it follows:

$$\tau_r(\delta, f) \leq C\delta \int_0^\delta t^{-2}t^\alpha\,dt = \mathcal{O}(\delta^\alpha).$$

Therefore, for $\alpha > 1$ and $f \in C_{2\pi}$ both moduli behave equivalent (cf. (3.1)):

$$\tau_r(\delta, f) = \mathcal{O}(\delta^\alpha) \Longleftrightarrow \omega_r(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha). \tag{3.5}$$

Regarding rates the averaged modulus does not show a disadvantage against the integral modulus. For $\alpha \leq 1$ one immediately gets a corresponding result for piecewise monotonous functions:

Lemma 3.1 *Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 2\pi$ and $f \in B_{2\pi}$ a function that is monotonous on each of the intervals $[x_{k-1}, x_k]$, $1 \leq k \leq n$. Then for $0 < \delta < \min_{1\leq k\leq n}(x_k - x_{k-1})$:*

$$\tau_1(\delta, f) \leq \omega_1(\delta, f, L^1_{2\pi}) + n\delta\omega_1(\delta, f).$$

Especially, for $0 < \alpha \leq 1$:

$$\omega_1(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha) \Longleftrightarrow \tau_1(\delta, f) = \mathcal{O}(\delta^\alpha). \tag{3.6}$$

Proof

$$\begin{aligned}\tau_1(\delta, f) &\leq n\delta\omega_1(\delta, f) + \sum_{k=1}^n \int_{x_{k-1}+\frac{\delta}{2}}^{x_k-\frac{\delta}{2}} \omega_1(\delta, f, t)\,dt \\ &\leq n\delta\omega_1(\delta, f) + \sum_{k=1}^n \int_{x_{k-1}+\frac{\delta}{2}}^{x_k-\frac{\delta}{2}} \left|f\left(t-\frac{\delta}{2}\right) - f\left(t+\frac{\delta}{2}\right)\right|\,dt \\ &\leq n\delta\omega_1(\delta, f) + \omega_1(\delta, f, L^1_{2\pi}).\end{aligned}$$

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In fact there are continuous functions f that are nowhere monotonous, i.e. there is no interval $[a,b] \subset [0,2\pi]$ on which f is monotonous. Therefore, it appears to be an open problem if one can replace piecewiese monotonicity by continuity, thus extending (3.5) to $\alpha > 0$.

Lemma 3.1 combined with (2.8) and (2.9), that hold true for both the integral and the averaged moduli, implies that

$$\begin{aligned} \omega_r(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha) &\iff \omega_1(\delta, f, L^1_{2\pi}) = \mathcal{O}(\delta^\alpha) \\ &\iff \tau_1(\delta, f) = \mathcal{O}(\delta^\alpha) \iff \tau_r(\delta, f) = \mathcal{O}(\delta^\alpha) \end{aligned}$$

for piecewise monotonous functions, $0 < \alpha < 1$, and $r \in \mathbb{N}$. So for many functions integral moduli and averaged moduli show the same rates.

4 An error bound for Fourier Laplace coefficients

After discussing properties of the τ-modulus we now apply this concept. We follow [14, p.41] to establish an error bound for $|f^\wedge(k) - (f^\wedge)^*_n(k)|$ and $f \in R_{2\pi}$ in terms of τ-moduli:

$$\begin{aligned} |f^\wedge(k)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(t)|\,dt \le \frac{1}{2\pi}\|f\|_{\frac{1}{n}}, \\ |(f^\wedge)^*_n(k)| &\le \frac{1}{2n+1}\sum_{j=0}^{2n} |f(u_{j,n})| = \frac{1}{2\pi}\sum_{j=0}^{2n}\int_{u_{j,n}}^{u_{j+1,n}} |f(u_{j,n})|\,dt \\ &\le \frac{1}{2\pi}\sum_{j=0}^{2n}\int_{u_{j,n}}^{u_{j+1,n}} M\left(f, t, \frac{2\pi}{2n+1}\right) dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} M\left(f, t, \frac{2\pi}{2n+1}\right) dt \\ &= \frac{1}{2\pi}\|f\|_{\frac{2\pi}{2n+1}} \le \frac{1}{2\pi}\|f\|_{\frac{\pi}{n}} \overset{(2.1)}{\le} \frac{1+\pi}{2\pi}\|f\|_{\frac{1}{n}}. \end{aligned}$$

Since $(f^\wedge)^*_n(k)$ is computed from discrete function values, $|(f^\wedge)^*_n(k)|$ can not be estimated against an integral norm of f.

To estimate the difference between $f^\wedge(k)$ and $(f^\wedge)^*_n(k)$ one uses the property that

$$p^\wedge(k) = (p^\wedge)^*_n(k)$$

for all $p \in \Pi_n$, i.e. trigonimetric polynomials p of degree at most n. This is because the interpolation polynomial of p equals to p. That allows us to compare $|f^\wedge(k) - (f^\wedge)^*_n(k)|$ with an error of best approximation. For each $p \in \Pi_n$ we get

$$\begin{aligned} |f^\wedge(k) - (f^\wedge)^*_n(k)| &= |[f-p]^\wedge(k) - ([f-p]^\wedge)^*_n(k)| \\ &\le |[f-p]^\wedge(k)| + |([f-p]^\wedge)^*_n(k)| \\ &\le \frac{1}{2\pi}\|f-p\|_{\frac{1}{n}} + \frac{1+\pi}{2\pi}\|f-p\|_{\frac{1}{n}} = \left[\frac{1}{2} + \frac{1}{\pi}\right]\|f-p\|_{\frac{1}{n}}. \end{aligned}$$

Therefore, the error E_n of best polynomial approximation is an upper bound:

$$|f^\wedge(k) - (f^\wedge)_n^*(k)| \leq \left[\frac{1}{2} + \frac{1}{\pi}\right] \inf\{\|f-p\|_{\frac{1}{n}} : p \in \Pi_n\} =: \left[\frac{1}{2} + \frac{1}{\pi}\right] E_n(f; R_{2\pi}). \quad (4.1)$$

Is is known (cf. [12]) that $E_n(f;R_{2\pi}) \leq C_r \tau_r(\frac{1}{n}, f)$ where the constant C_r is independent of f and n. We have shown the following result (see [14, p.41]):

Theorem 4.1 *For $f \in R_{2\pi}$ there holds true the error bound ($|k| \leq n$):*

$$|f^\wedge(k) - (f^\wedge)_n^*(k)| \leq \left[1 + \frac{1}{\pi}\right] C_r \tau_r\left(\frac{1}{n}, f\right) \leq [2+2\pi] C_r \omega_r\left(\frac{1}{n}, f\right). \quad (4.2)$$

With (3.5) this gives another proof of (1.1) for continuous functions. With (3.6) it extends (1.1) for piecewise monotonous functions like f_0 (see (1.4)) to $0 < \alpha \leq 1$.

5 Sharpness of the error bound

The main proposition of this article is the following result. It proves that in (4.2) the error can show the same rate as the τ-modulus, when at the same time the rate is strictly better than the rate of the uniform modulus (up to one power of $\frac{1}{n}$).

Theorem 5.1 *Let $0 < \alpha < 1$. For $r = 1$ and $0 < \beta < \alpha$, or $r \geq 2$ and $0 < \beta < 1 - \alpha$, there exists a counterexample $f_{\alpha,\beta} \in C_{2\pi}$ so that for each $k \in \mathbb{Z}$ there holds true*

$$\begin{aligned} \tau_r(\delta, f_{\alpha,\beta}) &= \mathcal{O}\left(\delta^{r-1+\alpha}\right), \\ \omega_r(\delta, f_{\alpha,\beta}) &\neq \begin{cases} \mathcal{O}(\delta^{\beta}), & r = 1, \\ \mathcal{O}(\delta^{r-2+\alpha+\beta}), & r \geq 2, \end{cases} \\ |f^\wedge_{\alpha,\beta}(k) - (f^\wedge_{\alpha,\beta})_n^*(k)| &\neq o\left(\frac{1}{n^{r-1+\alpha}}\right). \end{aligned}$$

Because of (4.1) the counterexample also establishes the sharpness of the estimate $E_n(f;R_{2\pi}) \leq C_r \tau_r(\frac{1}{n}, f)$ for the best approximation:

$$E_n(f_{\alpha,\beta}; R_{2\pi}) \neq o\left(\frac{1}{n^{r-1+\alpha}}\right).$$

As a consequence of (3.2) there is $\omega_r(\delta, f_{\alpha,\beta}) \leq \frac{4^{r+1}}{r\delta} \tau_r(\delta, f_{\alpha,\beta}) = \mathcal{O}(\delta^{r-2+\alpha})$. Therefore there can't be a counterexample with $\omega_r(\delta, f_{\alpha,\beta}) \neq \mathcal{O}(\delta^{r-2+\alpha})$.

For $r = 2$ the following proof additionally shows that one can set $\beta = 0$ if one restricts the values of k to a finite subset $\mathbb{F} \subset \mathbb{Z}$. Then there is a counterexample $f_{\alpha,0} \in C_{2\pi}$ such that for all $k \in \mathbb{F}$:

$$\begin{aligned} \tau_2(\delta, f_{\alpha,0}) &= \mathcal{O}\left(\delta^{1+\alpha}\right), \\ \omega_2(\delta, f_{\alpha,0}) &\geq c\delta^{\alpha}, & (5.1) \\ |f^\wedge_{\alpha,0}(k) - (f^\wedge_{\alpha,0})_n^*(k)| &\neq o\left(\frac{1}{n^{1+\alpha}}\right). & (5.2) \end{aligned}$$

In the context of Approximation Theory such negative results are often obtained on the basis of quantitative extensions of the uniform boundedness principle developed by Dickmeis, Nessel and van Wickeren (cf. [5] and [6]).

An abstract modulus of smoothness is a function ω, continuous on $[0,\infty)$ such that, for $0 < \delta_1, \delta_2$,

$$0 = \omega(0) < \omega(\delta_1) \leq \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2). \tag{5.3}$$

The functions $\omega(\delta) := \delta^\alpha$, $0 < \alpha \leq 1$, satisfy these conditions.

For a (real) Banach space X with norm $\|\cdot\|_X$ let $X^\sim$ be the set of non-negative-valued sublinear bounded functionals T on X, i.e., T maps X into $\mathbb{R}$ such that for all $f, g \in X$, $c \in \mathbb{R}$

$$Tf \geq 0, \qquad T(f+g) \leq Tf + Tg, \qquad T(cf) = |c|Ff,$$

$$\|T\|_{X^\sim} := \sup\{Tf : \|f\|_X \leq 1\} < \infty.$$

Theorem 5.2 *Suppose that for a family of remainders* $\{T_{n,k} : n \in \mathbb{N}, k \in \mathbb{B}_n\} \subset X^\sim$, *where* $(\mathbb{B}_n)_{n\in\mathbb{N}}$ *is a sequence of non-empty index sets, and for a measure of smoothness* $\{S_\delta : \delta > 0\} \subset X^\sim$ *there are test elements* $g_n \in X$ *such that* $(\delta > 0, n \to \infty)$

$$\begin{aligned}
\|g_n\|_X &\leq C_1 && \textit{for all } n \in \mathbb{N}, && (5.4)\\
S_\delta g_n &\leq C_2 \min\left\{1, \frac{\sigma(\delta)}{\varphi_n}\right\} && \textit{for all } n \in \mathbb{N},\ \delta > 0, && (5.5)\\
\|T_{n,k}\|_{X^\sim} &\leq C_{3,n} && \textit{for all } k \in \mathbb{B}_n,\ n \in \mathbb{N}, && (5.6)\\
T_{n,k} g_j &\leq C_{4,k} C_{5,j} \varphi_n && \textit{for all } 1 \leq j \leq n-1,\ k \in \mathbb{B}_n,\ n \in \mathbb{N}, && (5.7)\\
T_{n,k} g_n &\geq C_{6,k} > 0 && \textit{for all } k \in \mathbb{B}_n, && (5.8)
\end{aligned}$$

where $\sigma(\delta)$ *is a function, strictly positive on* $(0,\infty)$, *and* $(\varphi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *is a strictly decreasing sequence with* $\lim_{n\to\infty} \varphi_n = 0$. *Then for each modulus* ω *satisfying (5.3) and*

$$\lim_{\delta \to 0+} \frac{\omega(\delta)}{\delta} = \infty$$

there exists a (strictly increasing) subsequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ *and a counterexample* $f_\omega \in X$ *such that* $(\delta \to 0+,\ n \to \infty)$

$$\begin{aligned}
S_\delta f_\omega &= \mathcal{O}\left(\omega(\sigma(\delta))\right),\\
T_{n,k} f_\omega &\neq o(\omega(\varphi_n))
\end{aligned}$$

for each $k \in \mathbb{B} := \limsup_{k\to\infty} \mathbb{B}_{n_k} := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \mathbb{B}_{n_j}$.

For a proof using a gliding hump, further comments, and applications to Approximation Theory see [5, 9, 10] and the literature cited there. We use this general concept in the following proof.

Proof of Theorem 5.1 Let $r \in \mathbb{N}$. We choose parameters as follows: Let $X := C_{2\pi}^{(r-1)}$ be the Banach space with norm $\|f\|_{C_{2\pi}^{(r-1)}} := \sum_{k=0}^{r-1} \|f^{(k)}\|_{B_{2\pi}}$, $T_{n,k}f := n^{r-1}|f^\wedge(k) - (f^\wedge)_n^*(k)|$, $n \in \mathbb{N}$, $k \in \mathbb{B}_n := \{-n, -n+1, \ldots, n-1, n\}$ such that

$$T_{n,k}f \le n^{r-1}\left[\|f\|_{B_{2\pi}} + \frac{1}{2n+1}\sum_{j=0}^{2n}\|f\|_{B_{2\pi}}\right] \le n^{r-1}2\|f\|_{C_{2\pi}^{(r-1)}},$$

i.e., $\|T_{n,k}\|_{[C_{2\pi}^{(r-1)}]^\sim} \le 2\,n^{r-1}$ showing (5.6).

Setting $S_\delta f := \omega_1\left(\delta, f^{(r-1)}\right)$ it follows $S_\delta f \le 2\|f\|_{C_{2\pi}^{(r-1)}}$ and therefore $\|S_\delta\|_{[C_{2\pi}^{(r-1)}]^\sim} \le 2$, i.e. $S_\delta \in [C_{2\pi}^{(r-1)}]^\sim$.

The rate of convergence is described by $\sigma(\delta) := \delta$ and $\varphi_n := \frac{1}{n}$.

Key to the proof is the definition of the resonance sequence

$$g_n(t) := \frac{1}{n^{r-1}}\sum_{j=0}^{2n}\frac{1}{2^{|n-j|}}e^{i(n+1+j)t} \in \Pi_{3n+1}.$$

Please note that the lowest frequency of this sum is $n+1$ meaning $g_n^\wedge(k) = 0$, $|k| \le n$. When computing the error we therefore only have to deal with the Fourier Lagrange coefficients. The aliasing phenomenon will give the resonance condition (5.8).

We first verify (5.4):

$$\begin{aligned}\|g_n\|_{C_{2\pi}^{(r-1)}} &\le \frac{1}{n^{r-1}}\sum_{j=0}^{2n}\frac{1}{2^{|n-j|}}\|e^{i(n+1+j)t}\|_{C_{2\pi}^{(r-1)}} \\ &\le \frac{1}{n^{r-1}}\left[-1+2\sum_{j=0}^{n}\frac{1}{2^j}\right]\sum_{j=0}^{r-1}(3n+1)^j \le \left[-1+2\sum_{j=0}^{\infty}\frac{1}{2^j}\right]r\frac{(3n+1)^{r-1}}{n^{r-1}} \\ &= \left[-1+\frac{2}{1-\frac{1}{2}}\right]r\left[3+\frac{1}{n}\right]^{r-1} \le 3r\cdot 4^{r-1}. \end{aligned} \tag{5.9}$$

Condition (5.5) is satisfied because of

$$\begin{aligned} S_\delta g_n = \omega_1\left(\delta, g_n^{(r-1)}\right) &\le \delta\|g_n^{(r)}\|_{B_{2\pi}} \le \frac{\delta}{n^{r-1}}\left[\sum_{j=0}^{2n}\frac{1}{2^{|n-j|}}(n+1+j)^r\right] \\ &\le \frac{\delta}{n^{r-1}}\left[-1+\frac{2}{1-\frac{1}{2}}\right](3n+1)^r = \delta 3(3n+1)\left(\frac{3n+1}{n}\right)^{r-1} \\ &\le \delta 12n4^{r-1} = 4^{r-1}12\frac{\sigma(\delta)}{\varphi_n} \end{aligned}$$

and (see (5.9))

$$S_\delta g_n \le 2\|g_n^{(r-1)}\|_{B_{2\pi}} \le 2\|g_n\|_{C_{2\pi}^{(r-1)}} \le 6r4^{r-1}$$

so that $S_\delta g_n = \mathcal{O}(\min\{1, \sigma(\delta)/\varphi_n\})$.

To show (5.7) we discuss two cases: If $n \geq 3j+1$ Fourier- and Fourier Lagrange coefficients of $g_j \in \Pi_{3j+1}$ are identical: $T_{n,k}g_j = 0$. If $n < 3j+1$:

$$\begin{aligned} T_{n,k}g_j &\leq \|T_{n,k}\|_{[C_{2\pi}^{(r-1)}]^\sim}\|g_j\|_{C_{2\pi}^{(r-1)}} \leq 2\,n^{r-1}\cdot 3r\cdot 4^{r-1} \\ &\leq 2\cdot(3j+1)^{r-1}\cdot 3r\cdot 4^{r-1}\cdot\frac{3j+1}{n} = C_{5,j}\varphi_n. \end{aligned}$$

Together, both cases give (5.7):

$$T_{n,k}g_j \leq C_{5,j}\varphi_n \text{ for all } 1\leq j\leq n-1,\ k\in\mathbb{B}_n,\ n\in\mathbb{N}.$$

It remains to show the resonance condition (5.8). Because (1.1) holds true for g_n, we can compute the error:

$$\begin{aligned} T_{n,k}g_n &= n^{r-1}|g_n^\wedge(k)-(g_n^\wedge)_n^*(k)| = n^{r-1}|g_n^\wedge(k+2n+1)| \\ &= \frac{n^{r-1}}{n^{r-1}}\frac{1}{2^{|k|}} =: C_{6,k} > 0 \text{ for all } k\in\mathbb{B}_n. \end{aligned}$$

We can apply Theorem 5.2 for $\omega(\delta) := \delta^\alpha$. Since $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence it follows

$$\mathbb{B} := \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}\mathbb{B}_{n_j} = \bigcap_{k=1}^{\infty}\mathbb{Z} = \mathbb{Z}.$$

Therefore, we get a counterexample $f_\alpha \in C_{2\pi}^{(r-1)}$ such that for each $k\in\mathbb{Z}$ there holds true ($\delta\to 0+$, $n\to\infty$)

$$\begin{aligned} \omega_r(\delta,f_\alpha) &\overset{(2.4)}{\leq} \delta^{r-1}\omega_1\left(\delta, f_\alpha^{(r-1)}\right) = \delta^{r-1}S_\delta f_\alpha = \mathcal{O}(\delta^{r-1+\alpha}), \qquad (5.10) \\ \tau_r(\delta,f_\alpha) &\leq 2\pi\omega_r(\delta,f_\alpha) = \mathcal{O}\left(\delta^{r-1+\alpha}\right), \\ |f_\alpha^\wedge(k)-(f_\alpha^\wedge)_n^*(k)| &= \frac{1}{n^{r-1}}T_{n,k}f_\alpha \neq o\left(\frac{1}{n^{r-1+\alpha}}\right). \end{aligned}$$

We have shown the sharpness of the coarser error bound

$$|f^\wedge(k)-(f^\wedge)_n^*(k)| \leq C\omega_r\left(\frac{1}{n},f\right).$$

This especially is true for the finer bound $|f^\wedge(k)-(f^\wedge)_n^*(k)| \leq C\tau_r\left(\frac{1}{n},f\right)$. Now we will modify f_α by adding a function that is smooth with respect to the τ-modulus but less smooth with respect to the uniform modulus. Then only the finer bound becomes sharp.

Here we define 2π periodic functions $g_{\beta,1} := |\sin t|^{\frac{\beta}{2}}$, $g_{\beta,2} := |\sin t|^{\alpha+\frac{\beta}{2}}$, and for $r>2$

$$g_{\beta,r} := \int_0^t [g_{\beta,r-1} - g_{\beta,r-1}^\wedge(0)].$$

Because of (3.3) for $\alpha = \frac{\beta}{2}$ and (2.8), (2.7) we find ($r\geq 2$)

$$\tau_2(\delta,g_{\beta,1}) = \mathcal{O}\left(\delta^{1+\frac{\beta}{2}}\right) \Longrightarrow \tau_1(\delta,g_{\beta,1}) = \mathcal{O}(\delta),$$

$g_{\beta,r} \in C_{2\pi}^{(r-2)}$ and

$$\tau_r(\delta, g_{\beta,r}) \leq C_r \delta^{r-2} \tau_2(\delta, g_{\beta,2}) = \mathcal{O}\left(\delta^{r-1+\alpha+\frac{\beta}{2}}\right).$$

The function $f_{\alpha,\beta} := f_\alpha + g_{\beta,r}$ fulfills condition $\tau_r(\delta, f_{\alpha,\beta}) = \mathcal{O}(\delta^{r-1+\alpha})$.

The corresponding remainder for $r = 1$ is ($\frac{1}{n} = o\left(\frac{1}{n^\alpha}\right)$):

$$\begin{aligned} T_{n,k}\left(f_\alpha + g_{\beta,1}\right) &\geq T_{n,k} f_\alpha - T_{n,k} g_{\beta,1} \geq T_{n,k} f_\alpha - c_1 \tau_1\left(\frac{1}{n}, g_{\beta,1}\right) \\ &\geq T_{n,k} f_\alpha - c_2 \frac{1}{n} \neq o\left(\frac{1}{n^\alpha}\right). \end{aligned}$$

For $r \geq 2$ there is (note that $\frac{1}{n^{\alpha+\frac{\beta}{2}}} = o\left(\frac{1}{n^\alpha}\right)$):

$$T_{n,k}\left(f_\alpha + g_{\beta,r}\right) \geq T_{n,k} f_\alpha - T_{n,k} g_{\beta,r} \geq T_{n,k} f_\alpha - c_1 \frac{1}{n^{\alpha+\frac{\beta}{2}}} \neq o\left(\frac{1}{n^\alpha}\right).$$

If one sets $\beta = 0$ and restricts k to elements of $\mathbb{F} \subset \mathbb{Z}$, where $\mathbb{F}$ is a finite set, then this estimate also holds true for $f_{\alpha,0} := f_\alpha + \frac{c_0}{2c_1} g_{\beta,r}$ with

$$c_0 := \min_{k \in \mathbb{F}} \limsup_{n \to \infty} n^\alpha T_{n,k} f_\alpha > 0.$$

Especially for $r = 2$ this gives (5.2).

So $f_{\alpha,\beta}$ demonstrates that the error bound in terms of the averaged modulus is sharp. It remains to estimate the rate of the uniform modulus. For $r = 1$ ($0 < \delta < \frac{\pi}{2}$) and $\beta < \alpha$ there is

$$\begin{aligned} &\omega_1\left(\delta, f_\alpha + g_{\beta,1}\right) \geq \omega_1\left(\delta, g_{\beta,1}\right) - \omega_1\left(\delta, f_\alpha\right) \\ &\geq \left| |\sin(0+\delta)|^{\frac{\beta}{2}} - |\sin 0|^{\frac{\beta}{2}} \right| - c\delta^\alpha \geq \left(\frac{2}{\pi}\right)^{\frac{\beta}{2}} \delta^{\frac{\beta}{2}} - c\delta^\alpha \neq o\left(\delta^{\frac{\beta}{2}}\right). \end{aligned}$$

This implies $\omega_1\left(\delta, f_\alpha + g_{\beta,1}\right) \neq \mathcal{O}(\delta^\beta)$.

For $r \geq 2$ we start with (3.4), i.e.

$$\omega_2(\delta, g_{\beta,2}) \geq C\delta^{\alpha+\frac{\beta}{2}}. \tag{5.11}$$

Note that this is also true for $\beta = 0$ giving (5.1).

If we assume that $\omega_r(\delta, g_{\beta,r}) \leq C\delta^{r-2+\alpha+\beta}$, we get for $r > 2$:

$$\begin{aligned} \omega_{r-1}(\delta, g_{\beta,r-1}) = \omega_{r-1}(\delta, g'_{\beta,r}) &\overset{(2.5)}{\leq} C_1 \int_0^\delta \omega_r(t, g_{\beta,r}) \frac{1}{t^2}\, dt \\ &\leq C_2 \int_0^\delta t^{r-4+\alpha+\beta}\, dt \leq C_3 \delta^{(r-1)-2+\alpha+\beta}. \end{aligned}$$

By iterating this argument it follows

$$\omega_2(\delta, g_{\beta,2}) \leq C\delta^{\alpha+\beta}$$

in contradiction to (5.11). Because of (5.10) we have shown

$$\omega_r(\delta, f_\alpha + g_{\beta,r}) \geq \omega_r(\delta, g_{\beta,r}) - \omega_r(\delta, f_\alpha) \neq \mathcal{O}(\delta^{r-2+\alpha+\beta}).$$

■

So far we have excluded the case $\alpha = 1$. We restrict ourselves to $r = 2$ and again discuss

$$g(t) := g_{1,1}(t) = |\sin(t)| = \frac{1}{\pi}\left[2 + \sum_{k=1}^{\infty} \frac{1}{\frac{1}{4} - k^2}\cos 2kt\right] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{1}{\frac{1}{4} - k^2} e^{i2kt}.$$

We already know that (see (3.3) for $\alpha = 1$)

$$\tau_2(\delta, g) \leq 48\delta\omega_1(\delta, g) \leq 48\delta^2.$$

Let $|k| \leq n$. For coefficients with an even index we find:

$$\begin{aligned}
|(g^\wedge)^*_n(2k) - g^\wedge(2k)| &= \left|\sum_{m=1}^{\infty} g^\wedge(2k + m(2n+1)) + g^\wedge(2k - m(2n+1))\right| \\
&= \left|\sum_{m=1}^{\infty} g^\wedge(2k + 2m(2n+1)) + g^\wedge(2k - 2m(2n+1))\right| \\
&= \frac{1}{2\pi}\left|\sum_{m=1}^{\infty} \frac{1}{\frac{1}{4} - [k + m(2n+1)]^2} + \frac{1}{\frac{1}{4} - [k - m(2n+1)]^2}\right|.
\end{aligned}$$

All summands are negative and we can continue for $k > 0$

$$\begin{aligned}
|(g^\wedge)^*_n(2k) - g^\wedge(2k)| &\geq -\frac{1}{2\pi}\sum_{m=1}^{\infty} \frac{1}{\frac{1}{4} - [k + m(2n+1)]^2} \\
&\geq \frac{1}{2\pi}\sum_{m=1}^{\infty} \frac{1}{[k + m(2n+1)]^2} \geq \frac{1}{2\pi}\sum_{m=1}^{\infty} \frac{1}{[n + m(2n+1)]^2} \\
&\geq n^{-2}\frac{1}{2\pi}\sum_{m=1}^{\infty} \frac{1}{[1+3m]^2} = cn^{-2},
\end{aligned}$$

thus establishing the sharpness.

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