Convergence rates for Fourier partial sums of polygons and periodic splines

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Steffen Goebbels

Abstract Polygons can be seen as closed parameterized curves. Their parameterizations can be chosen as continuous, piecewise linear, periodic functions. Such functions possess a convergent Fourier series. Often polygons are classified with Fourier descriptors defined via Fourier coefficients of the parameterization. This fact motivates the discussion of the approximation error of Fourier partial sums of piecewise linear functions. More generally, the paper investigates convergence rates for periodic splines using elementary techniques of calculus. For example, such splines are used as curve parameterizations for active contours. Error bounds are shown to be best possible. An interesting effect is that the convergence rate at knots is different for odd and even degrees of piecewise polynomials. The slower rate for polynomials of odd degree can be used to detect dominant corners of contours.

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1 Introduction

In the field of pattern recognition, Fourier descriptors [5, 15] are standard features describing closed contours. They are derived from Fourier coefficients and often are invariant against scaling, rotation, translation and even shearing. [16] provides an overview and comparison.

Steffen Goebbels
Niederrhein University of Applied Sciences, Faculty of Electrical Engineering and Computer Science, Institute for Pattern Recognition
D-47805 Krefeld
e-mail: steffen.goebbels@hsnr.de

For practical purposes, closed contours have to be represented by few parameters based on sampled values. The most elementary approximation to a contour is by a polygon, i.e. by a curve with a piecewise linear parameterization that can be periodically continued. More generally, periodic splines can be used, see for example the monographs of Wahba [14, Chapter 2.1], Bojanov [3, pp. 117–131], Berlinet and Thomas-Agnan [2, Chapter 2.4.2], and de Boor [6, pp. 282–289]. For segmenting or reproducing shapes, active contours based on periodic splines (also known as snake splines) are an established means (see [10], [7] gives an overview).

Fourier descriptors often have to be computed on such representations because the original exact contours are not available due to sampling. Therefore it becomes interesting to analyze Fourier series of piecewise linear functions, and more generally, of periodic splines. The better Fourier partial sums represent such splines, the better Fourier descriptors might represent a corresponding contour.

Spline-based quadrature formulas are used to improve the numerical computation of Fourier coefficients, see [6, pp. 288–289] and the literature cited there. However, we start with the spline representation and compute its exact Fourier coefficients. Those can be seen as numerically computed Fourier coefficients of an unknown, exact parameterization of a contour curve. Given the exact spline's Fourier coefficients, we analyze error bounds for Fourier partial sum approximations of the splines.

Whereas it might be difficult to exactly compute Fourier coefficients of parameterization functions of arbitrary curves, they can easily be obtained for splines (cf. [13] for polygons), especially if equidistant knots are used (cf. [2, p. 122]). This allows us to give uniform and point-wise error estimates for the approximation with finite Fourier partial sums in Section 3. The paper also discusses sharpness of these estimates in Section 4. It is somewhat surprising that partial sums of splines with even polynomial degree show a higher rate of convergence at knots than partial sums of splines with odd polynomial degree. The paper concludes with an application: The approximation error of odd degree splines is used to detect dominant corners.

2 Complex-valued periodic spline parameterizations

Our aim is to discuss approximations of parameterized closed curves in the complex plane (see Figures 1 and 2). The set of points of a curve can be obtained by an infinite number of parameterizations (c.f. the discussion in [6, p. 278]). We restrict ourselves to complex-valued parameterization functions f for which real part $\mathrm{Re}(f)$ and imaginary part $\mathrm{Im}(f)$ are real valued, 2π-periodic splines with pairwise different knots $t_0,\ldots,t_{m-1} \in [0,2\pi)$, $t_m := t_0 + 2\pi$, $t_{-1} := t_{m-1} - 2\pi$. Thus f is a complex-valued, periodic spline that can be written with polynomials that have complex coefficients but a real variable. Between knots, the real-valued component splines are polynomials of degree s. We require $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ to be $s-1$-times continuously differentiable. For the computation of lower error bounds, we also assume that at each knot $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ is not s-times differentiable, i.e. the complex-valued spline is not s-times differentiable there.

Polygons with vertices (x_0, y_0), (x_1, y_1), $\ldots$, $(x_m, y_m) = (x_0, y_0)$, $m \geq 3$, can be parameterized with piecewise linear splines of degree $s = 1$. For convenience, we extend the set of vertices m-periodically, i.e. $(x_{r+m}, y_{r+m}) = (x_r, y_r)$. We require that no vertex lies on a straight line going through its predecessor and successor. Thus, subsequent vertices have to be different. If these vertices are the spline's values at knots then the spline is not differentiable at the knots.

For example, a parameterization f of the polygon can be obtained by traveling along the curve with constant speed or by traveling along each edge within the same time so that speed for longer edges is faster but constant per edge.

For obtaining a parameterization with constant speed, let

$$l_r := \sqrt{(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2} > 0$$

be the length of the r-th edge and

$$d_0 := 0, \quad d_r := d_{r-1} + l_r,\ 1 \leq r \leq m,$$

be the distance from the start point (x_0, y_0) to (x_r, y_r) along the edges of the polygon. Then we define knots

$$t_r := \frac{2\pi}{d_m} d_r,\ 0 \leq r \leq m,$$

and $t_{-1} := t_{m-1} - t_m = t_{m-1} - 2\pi$. The argument of f is the arc length, normalized to an overall length of 2π. A constant time per edge parameterization can be obtained with equally spaced knots

$$t_r := r\frac{2\pi}{m}.$$

In both cases, the polygon is parametrized for $0 \leq t < 2\pi$ with the piecewise linear, 2π-periodic spline ($1 \leq r \leq m$, i denotes the imaginary unit)

$$f(t) := x_{r-1} + iy_{r-1} + \frac{[x_r + iy_r] - [x_{r-1} + iy_{r-1}]}{t_r - t_{r-1}}(t - t_{r-1}) \text{ for } t_{r-1} \leq t < t_r.$$

A periodic, complex-valued spline f can be represented by a point-wise convergent Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

with Fourier coefficients $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt$. We restrict ourselves to complex valued functions that parameterize curves in 2D. However, everything can be transferred to higher dimensions if one separately computes Fourier series of each component function.

3 Error bounds

If one replaces the infinite Fourier sum by a symmetric finite Fourier partial sum $\sum_{k=-M}^{M} c_k e^{ikt}$, then higher frequencies are being cut off. The result is a low pass filtered version of f that lacks detail, see Figures 1 and 2. We discuss splines f that are $s-1$-times continuously differentiable such that $f^{(s-1)}$ is Lipschitz-continuous. For such functions the Jackson-type estimate

$$\max\{|R(t,M)| : t \in \mathbb{R}\} \in O\left(\frac{\log(M)}{M^s}\right)$$

for the error

$$R(t,M) := f(t) - \sum_{k=-M}^{M} c_k e^{ikt}$$

is well known (see [11, S. 136]). The log-factor originates from the L^1-norm of the Dirichlet kernel. However, due to the restriction to splines, we can easily prove better uniform and point-wise estimates without log-factor using elementary calculus. The main tool will be partial summation.

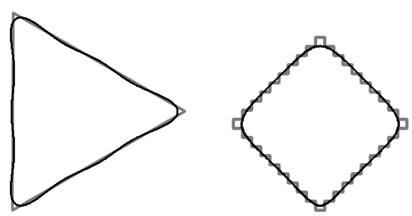

Fig. 1 Left: Fourier partial sum $\sum_{k=-2}^{2} c_k e^{ikt}$ of triangle. Right: Fourier partial sum $\sum_{k=-4}^{4} c_k e^{ikt}$ of noisy square

Theorem 1. *Let $f_{(s)}$ be a $s-1$-times continuously differentiable, 2π-periodic spline consisting of piecewise polynomials of degree s with (different) knots t_r, $1 \leq r \leq m$, such that $f_{(1)} := f_{(s)}^{(s-1)}$ is not differentiable at each knot. Then (with standard Landau symbols O and Ω)*

$$\max\{|R(t,M)| : t \in \mathbb{R}\} \in O\left(\frac{1}{M^s}\right), \tag{1}$$

and this order of uniform convergence exactly holds at the knots t_r if s is odd:

$$|R(t_r,M)| \in \Omega\left(\frac{1}{M^s}\right). \tag{2}$$

For odd s, the convergence rate at knots t_r is bounded by

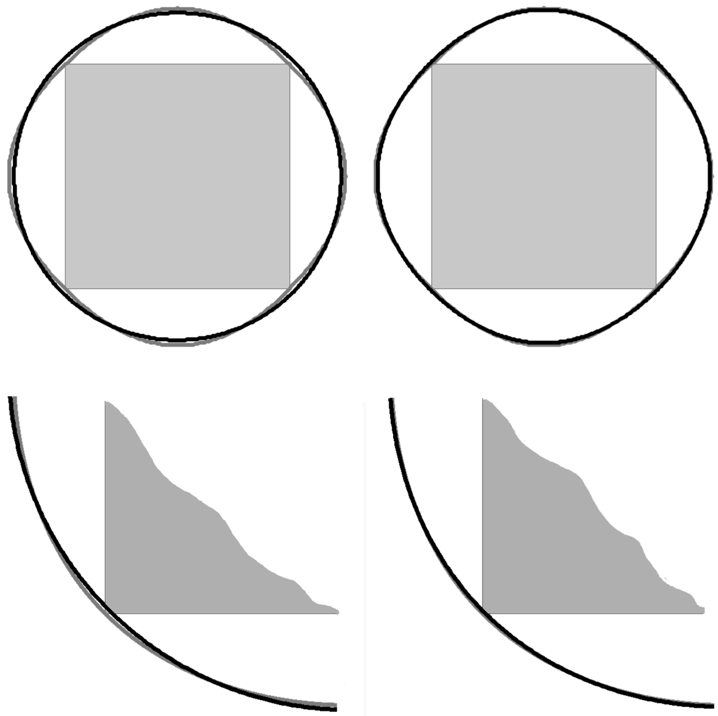

Fig. 2 The closed grey curves in the upper two pictures are drawn using a periodic spline parameterization. The spline has degree $s = 2$ and $m = 4$ equidistant knots at which the vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ of a square are reproduced. The lower two pictures show a section of a curve in grey. This curve is parameterized by a spline of degree $s = 3$ and the same $m = 4$ knots. The black curves are approximations by Fourier partial sums $\sum_{k=-2}^{2} c_k e^{ikt}$ (left column) and $\sum_{k=-4}^{4} c_k e^{ikt}$ (right column), respectively.

$$\frac{|\alpha_r|}{\pi}\frac{1}{(M+1)^s} - \frac{C}{M^{s+1}} \le |R(t_r, M)| \le \frac{|\alpha_r|}{\pi}\frac{1}{M^s} + \frac{C}{M^{s+1}} \tag{3}$$

where the constant C is independent of M, and

$$\alpha_r := \frac{f_{(1)}(t_r) - f_{(1)}(t_{r-1})}{t_r - t_{r-1}} - \frac{f_{(1)}(t_{r+1}) - f_{(1)}(t_r)}{t_{r+1} - t_r} \neq 0. \tag{4}$$

Especially, if knots are chosen equidistantly ($t_r = r\frac{2\pi}{m}$) then the numerator of α_r is a second difference:

$$\alpha_r = -\frac{f_{(1)}(t_{r-1}) - 2f_{(1)}(t_r) + f_{(1)}(t_{r+1})}{\frac{2\pi}{m}}. \tag{5}$$

For all values of $t \in [0,2\pi)$, $t \neq t_r$ for $0 \le r < m$, convergence is faster:

$$|R(t,M)| \le \frac{2^{s+1}}{\pi(s+1)} \left[\sum_{r=0}^{m-1} |\alpha_r| \frac{|t-t_r|}{1-\cos(t-t_r)} \right] \frac{1}{M^{s+1}}. \tag{6}$$

If s is even, then we also obtain this rate of convergence for knots $t = t_{r_0}$:

$$\left|R(t_{r_0},M)\right| \le \frac{2^{s+1}}{\pi(s+1)} \left[\sum_{r=0, r\neq r_0}^{m-1} |\alpha_r| \frac{|t_{r_0}-t_r|}{1-\cos(t_{r_0}-t_r)} \right] \frac{1}{M^{s+1}}. \tag{7}$$

Since $f_{(1)}$ has to be not-differentiable at the knots t_r, this function is a 2π-periodic, piecewise linear parameterization of a polygon for which no vertex lies on a straight line through its neighbors.

A Riemann-Lebesgue lemma with orders (see [4, p. 168]) shows that spline's Fourier coefficients can be asymptotically bounded by $\frac{1}{M^{s+1}}$ which in turn yields (1). However, the other estimates of the theorem need a different proof.

In the next section we present examples for which the rate $\frac{1}{M^{s+1}}$ is best possible on a dense set.

Proof. For convenience, we use a basis of 2π-periodic hat functions to represent piecewise linear function $f_{(1)} = f_{(s)}^{(s-1)}$. Thus, $f_0, \ldots, f_{m-1}$ are defined on $[t_{r-1}, t_{r-1}+2\pi)$ via

$$f_r(t) := \begin{cases} \frac{t-t_{r-1}}{t_r-t_{r-1}}, & t_{r-1} \le t < t_r \\ \frac{t_{r+1}-t}{t_{r+1}-t_r}, & t_r \le t < t_{r+1} \\ 0, & t_{r+1} \le t < t_{r-1}+2\pi. \end{cases}$$

Then $f_{(1)}$ can be written as $f_{(1)}(t) = \sum_{r=0}^{m-1} f_{(1)}(t_r) f_r(t)$. With

$$c_{k,r} := \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(t) e^{-ikt}\, dt$$

we denote Fourier coefficients of f_r. Then $c_{0,r} = \frac{t_{r+1}-t_{r-1}}{4\pi}$. Using partial integration, it is easy to compute $c_{k,r}$ for $0 \le r < m$ and $k \neq 0$:

$$\begin{aligned} c_{k,r} &= \frac{1}{2\pi} \int_{t_{r-1}}^{t_r} \frac{t-t_{r-1}}{t_r-t_{r-1}} e^{-ikt}\, dt - \frac{1}{2\pi} \int_{t_r}^{t_{r+1}} \frac{t-t_{r+1}}{t_{r+1}-t_r} e^{-ikt}\, dt \\ &= \frac{1}{2\pi} \left[\frac{i}{k} e^{-ikt_r} + \frac{1}{k^2} \frac{e^{-ikt_r} - e^{-ikt_{r-1}}}{t_r - t_{r-1}} \right] - \frac{1}{2\pi} \left[\frac{i}{k} e^{-ikt_r} + \frac{1}{k^2} \frac{e^{-ikt_{r+1}} - e^{-ikt_r}}{t_{r+1}-t_r} \right] \\ &= \frac{1}{2\pi} \frac{1}{k^2} \left[\frac{e^{-ikt_r} - e^{-ikt_{r-1}}}{t_r - t_{r-1}} - \frac{e^{-ikt_{r+1}} - e^{-ikt_r}}{t_{r+1}-t_r} \right]. \end{aligned}$$

Fourier coefficients of $f_{(1)}$ are ($k \neq 0$)

$$
\begin{aligned}
c_k &= \sum_{r=0}^{m-1} f_{(1)}(t_r) c_{k,r} \\
&= \frac{1}{2\pi}\frac{1}{k^2}\sum_{r=0}^{m-1} f_{(1)}(t_r)\left[\frac{e^{-ikt_r}-e^{-ikt_{r-1}}}{t_r-t_{r-1}} - \frac{e^{-ikt_{r+1}}-e^{-ikt_r}}{t_{r+1}-t_r}\right] \\
&= \frac{1}{2\pi}\frac{1}{k^2}\left[-\sum_{r=-1}^{m-2} f_{(1)}(t_{r+1})\frac{e^{-ikt_r}}{t_{r+1}-t_r}\right. \\
&\quad \left. + \sum_{r=0}^{m-1}\left(\frac{f_{(1)}(t_r)e^{-ikt_r}}{t_r-t_{r-1}} + \frac{f_{(1)}(t_r)e^{-ikt_r}}{t_{r+1}-t_r}\right) - \sum_{r=1}^{m} f_{(1)}(t_{r-1})\frac{e^{-ikt_r}}{t_r-t_{r-1}}\right] \\
&= \frac{1}{2\pi}\frac{1}{k^2}\sum_{r=0}^{m-1}\alpha_r e^{-ikt_r} \qquad (8)
\end{aligned}
$$

with α_r as given in (4) due to 2π-periodicity of the functions and $t_{-1} = t_{m-1} - 2\pi$.

Because $f_{(1)}$ is not differentiable at the knots, all

$$\alpha_r \neq 0. \qquad (9)$$

For $k \neq 0$, iterative calculation of antiderivatives of $f_{(1)}$ to $f_{(s)}$ results in Fourier coefficients $\frac{c_k}{(ik)^{s-1}}$ of $f_{(s)}$, $k \neq 0$ (see [4, p. 172]). Thus we have to estimate the error

$$
\begin{aligned}
|R(t,M)| &= \left|\sum_{k=M+1}^{\infty}\frac{c_{-k}}{(-ik)^{s-1}}e^{-ikt} + \frac{c_k}{(ik)^{s-1}}e^{ikt}\right| \\
&= \frac{1}{2\pi}\left|\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}}\sum_{r=0}^{m-1}\alpha_r[(-1)^{s-1}e^{-ik(t-t_r)} + e^{ik(t-t_r)}]\right| \qquad (10)
\end{aligned}
$$

of approximating $f_{(s)}$ with a Fourier partial sum. Obviously,

$$|R(t,M)| \leq \frac{\sum_{r=0}^{m-1}|\alpha_r|}{\pi}\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}} \in O\left(\frac{1}{M^s}\right),$$

since $\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}} \leq \int_M^{\infty}\frac{1}{x^{s+1}}\,dx = \frac{1}{M^s}$. This proves the uniform upper bound (1).

To show a higher rate of convergence at points that are different from knots, we use Abel's partial summation (cf. [4, p. 51]).

Lemma 1. *Let $\varphi \neq k2\pi$ for all $k \in \mathbb{Z}$ and $(a_k)_{k=1}^{\infty}$ be a sequence with*

$$\lim_{k\to\infty} a_k = 0 \text{ and } \sum_{k=1}^{\infty}|a_k - a_{k+1}| < \infty.$$

For $M \in \mathbb{N}$ let $R_M := \sum_{k=M+1}^{\infty} a_k e^{i\varphi k}$, then

$$R_M = \sum_{k=M+1}^{\infty} \frac{(e^{i\varphi})^{M+1} - (e^{i\varphi})^{k+1}}{1-e^{i\varphi}}[a_k - a_{k+1}], \tag{11}$$

$$|R_M| \le \frac{2}{|1-e^{i\varphi}|} \sum_{k=M+1}^{\infty} |a_k - a_{k+1}|. \tag{12}$$

Proof. Since $e^{i\varphi} \neq 1$, we can compute the following geometric sum for $l > M$: $\sum_{k=M+1}^{l}(e^{i\varphi})^k = \frac{(e^{i\varphi})^{M+1}-(e^{i\varphi})^{l+1}}{1-e^{i\varphi}}$. Using Abel's partial summation, we obtain

$$\begin{aligned}\sum_{k=M+1}^{s} a_k (e^{i\varphi})^k &= a_s \sum_{k=M+1}^{s} (e^{i\varphi})^k + \sum_{k=M+1}^{s-1}\left[\sum_{j=M+1}^{k}(e^{i\varphi})^j\right][a_k - a_{k+1}]\\
&= a_s \frac{(e^{i\varphi})^{M+1} - (e^{i\varphi})^{s+1}}{1-e^{i\varphi}} + \sum_{k=M+1}^{s-1} \frac{(e^{i\varphi})^{M+1} - (e^{i\varphi})^{k+1}}{1-e^{i\varphi}}[a_k - a_{k+1}].\end{aligned}$$

The limit $s \to \infty$ gives (11). This directly implies the upper estimate (12) because $|(e^{i\varphi})^{M+1} - (e^{i\varphi})^{k+1}| \le 2$. $\square$

To continue the proof of Theorem 1, we apply (11) to sums of the following type:

$$\begin{aligned}&\left|\sum_{k=M+1}^{\infty} a_k[(-1)^{s-1}e^{-i\varphi k} + e^{i\varphi k}]\right|\\
&\le \sum_{k=M+1}^{\infty}\Big|(-1)^{s-1}[1-e^{i\varphi}][e^{-i\varphi(M+1)} - e^{-i\varphi(k+1)}]\\
&\qquad\qquad +[1-e^{-i\varphi}][e^{i\varphi(M+1)} - e^{i\varphi(k+1)}]\Big| \frac{|a_k - a_{k+1}|}{|[1-e^{i\varphi}][1-e^{-i\varphi}]|}\\
&= \sum_{k=M+1}^{\infty}\Big|(-1)^{s-1}e^{-i\varphi(M+1)} + e^{i\varphi(M+1)} - (-1)^{s-1}e^{-i\varphi(k+1)} - e^{i\varphi(k+1)}\\
&\qquad\qquad -(-1)^{s-1}e^{-i\varphi M+} - e^{i\varphi M} + (-1)^{s-1}e^{-i\varphi k} + e^{i\varphi k}\Big| \frac{|a_k - a_{k+1}|}{2-2\cos(\varphi)}.\end{aligned}$$

If s is even, the left factor in the sum is

$$2|\sin(\varphi(M+1)) - \sin(\varphi M) - \sin(\varphi(k+1))) + \sin(\varphi k)|,$$

for odd s we get

$$2|\cos(\varphi(M+1)) - \cos(\varphi M) - \cos(\varphi(k+1))) + \cos(\varphi k)|.$$

By applying the mean value theorem to the differences, we obtain in both cases

$$\left|\sum_{k=M+1}^{\infty} a_k[(-1)^{s-1}e^{-i\varphi k} + e^{i\varphi k}]\right| \le \frac{2|\varphi|}{1-\cos(\varphi)} \sum_{k=M+1}^{\infty} |a_k - a_{k+1}|. \tag{13}$$

If we choose $a_k = \frac{1}{k^{s+1}}$ then (13) becomes

$$\begin{aligned} &\left|\sum_{k=M+1}^{\infty} \frac{1}{k^{s+1}}[(-1)^{s-1}e^{-i\varphi k} + e^{i\varphi k}]\right| \le \frac{2|\varphi|}{1-\cos(\varphi)} \sum_{k=M+1}^{\infty} \frac{\sum_{\lambda=0}^{s}\binom{s+1}{\lambda}k^{\lambda}}{k^{s+1}(k+1)^{s+1}} \\ &\le \frac{2^{s+2}|\varphi|}{1-\cos(\varphi)} \sum_{k=M+1}^{\infty} \frac{1}{k^{2s+2-s}} \le \frac{2^{s+2}|\varphi|}{1-\cos(\varphi)} \int_M^{\infty} \frac{1}{x^{s+2}}\,dx = \frac{2^{s+2}|\varphi|}{1-\cos(\varphi)} \frac{\frac{1}{s+1}}{M^{s+1}}. \end{aligned} \tag{14}$$

We apply this to (10), compute a common denominator and obtain (6). For even s this also gives (7) because for $t = t_r$ the r-th summand in (10) results in zero due to

$$(-1)^{s-1}e^{-ik(t-t_r)} + e^{ik(t-t_r)} = -1+1 = 0.$$

This is the reason why partial sums of periodic splines of even degree have a higher convergence rate at knots. The other summands can be estimated as previously done.

Using (6), we can also prove (2) for odd s: Let $t = t_{r_0}$ for $0 \le r_0 < m$. According to (10) we estimate the error $|R(t_{r_0},M)|$ by $|R(t_{r_0},M)| \ge S_1 - S_2$ with

$$S_1 := \left|\frac{1}{2\pi}\sum_{k=M+1}^{\infty} \frac{\alpha_{r_0}}{k^{s+1}}[(-1)^{s-1}e^{-ik(t_{r_0}-t_{r_0})} + e^{ik(t_{r_0}-t_{r_0})}]\right| = \frac{|\alpha_{r_0}|}{\pi}\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}},$$

where $\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}} \ge \int_{M+1}^{\infty}\frac{1}{x^{s+1}} = \frac{s}{(M+1)^s} \ge \frac{s}{(2M)^s}$ and $|\alpha_{r_0}| > 0$, see (9). On the other side, equation (14) for $\varphi := t_{r_0} - t_r$ shows that

$$\begin{aligned} S_2 &:= \left|\frac{1}{2\pi}\sum_{r\in\{0,\dots,m-1\}\setminus\{r_0\}} \alpha_r \sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}}[(-1)^{s-1}e^{-ik(t_{r_0}-t_r)} + e^{ik(t_{r_0}-t_r)}]\right| \\ &\le \frac{1}{M^{s+1}}\cdot\frac{1}{2\pi}\sum_{r\in\{0,\dots,m-1\}\setminus\{r_0\}} |\alpha_r| \frac{2^{s+2}|t_{r_0}-t_r|\frac{1}{s+1}}{1-\cos(t_{r_0}-t_r)} =: \frac{C}{M^{s+1}} \end{aligned}$$

so that the error has a lower bound

$$|R(t_{r_0},M)| \ge \frac{|\alpha_{r_0}|s}{2^s\pi}\frac{1}{M^s} - \frac{C}{M^{s+1}},$$

i. e. we have shown (2). For odd s, we especially have $S_1 - S_2 \le |R(t_{r_0},M)| \le S_1 + S_2$ with

$$\frac{|\alpha_{r_0}|}{\pi}\frac{1}{(M+2)^s} \le S_1 := \frac{|\alpha_{r_0}|}{\pi}\sum_{k=M+1}^{\infty}\frac{1}{k^{s+1}} \le \frac{|\alpha_{r_0}|}{\pi}\frac{1}{M^s}$$

and $S_2 \le \frac{C}{M^{s+1}}$ so that, completing the proof of Theorem 1, (3) follows. $\square$

4 Sharpness

Theorem 1 provides uniform error bounds with order $1/M^s$. But (6) cannot be used to obtain a uniform bound with rate $1/M^{s+1}$ because

$$\lim_{t\to t_r}\frac{|t-t_r|}{1-\cos(t-t_r)}=\infty.$$

In fact, estimate (6) (including (7) for even s) cannot be improved to become a uniform error bound with order $1/M^{s+1}$:

Theorem 2. *Under the assumptions of Theorem 1, a uniform bound*

$$\max\{|R(t,M)|: t\in[0,2\pi), t\neq t_r, 0\leq r<m\}\leq\frac{C_0}{M^{s+\alpha}}$$

with a constant C_0 independent of M does not hold true for any $\alpha>0$.

Proof. For odd s, the proof is obvious, because for each M the error $R(t,M)$ is continuous in t: For $\varepsilon=1/M^{s+\alpha}$ and a knot t_r there exists a t_ε (that is not a knot) with $|R(t_r,M)-R(t_\varepsilon,M)|\leq\frac{1}{M^{s+\alpha}}$. If we assume a uniform bound to exist, then we can apply it for $t=t_\varepsilon$ and get

$$|R(t_r,M)|\leq|R(t_\varepsilon,M)|+|R(t_r,M)-R(t_\varepsilon,M)|\leq\frac{C_0+1}{M^{s+\alpha}},$$

which contradicts (2).

The non-existence of a uniform bound for odd s also implies the non-existence of such a bound for even s. Without restriction let $0<\alpha<2$. For given $f_{(s)}$, s even, we define

$$f_{(s+1)}(t):=-t\frac{\int_0^{2\pi}f^{(s)}(u)\,du}{2\pi}+\int_0^t f_{(s)}(u)\,du$$

as a spline of odd degree $s+1$, i.e. $f'_{(s+1)}=f_{(s)}-\frac{1}{2\pi}\int_0^{2\pi}f^{(s)}(u)\,du$. With $R(t,M,f_{(s)})$ and $R(t,M,f_{(s+1)})$ we denote the partial sum error functionals of the two splines. Then

$$\begin{aligned}&\frac{d}{dt}R(t,M,f_{(s+1)})=f'_{(s+1)}(t)-\sum_{k=-M}^{M}ikc_ke^{ikt}\\&=f_{(s)}(t)-\frac{1}{2\pi}\int_0^{2\pi}f^{(s)}(u)\,du-\sum_{k=-M,k\neq0}^{M}ikc_ke^{ikt}=R(t,M,f_{(s)}),\end{aligned}\tag{15}$$

where c_k are Fourier coefficients of $f_{(s+1)}$, and ikc_k, $k\neq0$, are Fourier coefficients of $f_{(s)}$. We assume that a uniform error bound

$$\max\left\{\left|R(t,M,f_{(s)})\right|: t\in[0,2\pi), t\neq t_r, 0\leq r<m\right\}\leq\frac{C_0}{M^{s+\alpha}}$$

holds and contradict this assumption. For $t_M \in \left(t_r - \frac{t_r - t_{r-1}}{2}, t_r + \frac{t_{r+1} - t_r}{2}\right) \setminus \{t_r\}$ with $|t_r - t_M| < 1/M^{1-\frac{\alpha}{2}}$ there exists a ξ between t_r and t_M such that (see (6), (15))

$$\begin{aligned}
&|R(t_r, M, f_{(s+1)})| \leq |R(t_M, M, f_{(s+1)})| + |R(t_r, M, f_{(s+1)}) - R(t_M, M, f_{(s+1)})| \\
&\leq \frac{C_1}{M^{s+2}} + \frac{|t_M - t_r|}{1 - \cos(t_M - t_r)} \frac{C_2}{M^{s+2}} + |t_M - t_r| \left|R'(\xi, M, f_{(s+1)})\right| \\
&\leq \frac{C_1}{M^{s+2}} + \frac{\frac{1}{M^{1-\frac{\alpha}{2}}}}{1 - \cos(t_M - t_r)} \frac{C_2}{M^{s+2}} + \frac{1}{M^{1-\frac{\alpha}{2}}} \left|R(\xi, M, f_{(s)})\right| \\
&\leq \frac{C_1}{M^{s+2}} + \frac{1}{1 - \cos(t_M - t_r)} \frac{C_2}{M^{s+3-\frac{\alpha}{2}}} + \frac{1}{M^{1-\frac{\alpha}{2}}} \frac{C_0}{M^{s+\alpha}}.
\end{aligned}$$

Taylor expansion of $1 - \cos(t_M - t_r)$ has a lowest order term $\frac{1}{2}(t_M - t_r)^2$ so that $\frac{1}{1-\cos(t_M - t_r)} \in O(M^{2-\alpha})$. Thus $|R(t_r, M, f_{(s+1)})|$ is bounded by $C_3/M^{s+1+\frac{\alpha}{2}}$ uniformly. This contradicts (2) for degree $s+1$. $\square$

The rest of this section deals with construction of counter examples to show that the point-wise convergence rate $\frac{1}{M^{s+1}}$ of Theorem 1 is best possible. A standard technique in Approximation Theory is to prove existence of counter examples with the uniform boundedness principle (see [8]). However, this requires a Banach space. Therefore, the approach is not suitable for our simple function spaces of splines. We explicitly define counter examples based on regular polygons. Compared with arbitrary periodic splines, convergence rates of their Fourier partial sums should be quite good because regular polygons approximate the unit circle. But it turns out, that rates are not better than $\frac{1}{M^{s+1}}$.

Lemma 2. *Let $(R_M)_{M=1}^{\infty}$ be a sequence of continuous, real functions $R_M : [a,b] \to \mathbb{R}$ and D a dense set in $[a,b]$. If a common constant $c > 0$ exists such that*

$$\limsup_{M\to\infty} R_M(t) > c \text{ for each } t \in D$$

then

$$\limsup_{M\to\infty} R_M(t) \geq c \text{ for each } t \in [a,b].$$

Proof. Let $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < c$. For each $u \in D$ a (potentially different) strictly increasing sequence $(M_{u,j})_{j=1}^{\infty} \subset \mathbb{N}$ exists such that $R_{M_{u,j}}(u) > c$ for all $j \in \mathbb{N}$. Due to the continuity of each $R_{M_{u,j}}$ there exist $\delta_{M_{u,j}} > 0$ such that

$$|R_{M_{u,j}}(t) - R_{M_{u,j}}(u)| < \varepsilon \text{ for all } t \in (u - \delta_{M_{u,j}}, u + \delta_{M_{u,j}}) \cap [a,b],$$

$$R_{M_{u,j}}(t) \geq R_{M_{u,j}}(u) - |R_{M_{u,j}}(t) - R_{M_{u,j}}(u)| \geq c - \varepsilon. \tag{16}$$

For each $t \in [a,b]$ we iteratively construct $(\tilde{M}_{t,n})_{n=1}^{\infty} \subset \mathbb{N}$ as a strictly increasing sequence such that

$$R_{\tilde{M}_{t,n}}(t) > c - \varepsilon. \tag{17}$$

Since $\varepsilon > 0$ can be chosen arbitrarily, this proves the lemma. We use a helper sequence $(a_n)_{n=1}^{\infty}$ starting with $a_1 := 0$. Now we iteratively define next elements. The n-th step of the iteration (which begins with $n = 1$) constructs $\tilde{M}_{t,n}$ for each t and defines a_{n+1}:

For each $u \in D$ the sequence $(M_{u,j})_{j=1}^{\infty} \subset \mathbb{N}$ is strictly increasing so that there exists a $j_{u,n} \in \mathbb{N}$ with $M_{u,j_{u,n}} > a_n$ and $R_{M_{t,j_{u,n}}}(t) \geq c - \varepsilon$ for all $t \in (u - \delta_{M_{u,j_{u,n}}}, u + \delta_{M_{u,j_{u,n}}}) \cap [a,b]$. The open sets $(u - \delta_{M_{u,j_{u,n}}}, u + \delta_{M_{u,j_{u,n}}})$, $u \in D$, are an open cover of $[a,b]$. According to the theorem of Heine and Borel, a finite subcover $U_1 := (u_1 - \delta_{M_{u_1,j_{u_1},n}}, u_1 + \delta_{M_{u_1,j_{u_1},n}}), \ldots, U_{k_n} := (u_{k_n} - \delta_{M_{u_{k_n},j_{u_{k_n}},n}}, u_{k_n} + \delta_{M_{u_{k_n},j_{u_{k_n}},n}})$ exists. Each $t \in [a,b]$ is element of (at least) one of these intervals. Let U_l be the first interval (in the order of the previous list) that covers t. Then we set $\tilde{M}_{t,n} := M_{u_l,j_{u_l},n}$ and according to (16) we have $R_{\tilde{M}_{t,n}}(t) > c - \varepsilon$. Thus we have selected suitable $\tilde{M}_{t,n}$ for each $t \in [a,b]$.

For the next iteration we define $a_{n+1} := \max\{M_{u_l,j_{u_l},n} : 1 \leq l \leq k_n\}$. This ensures that in the next iteration $\tilde{M}_{t,n+1} > \tilde{M}_{t,n}$ will be selected for each $t \in [a,b]$. We get strictly increasing sequences $(\tilde{M}_{t,n})_{n=1}^{\infty}$ that fulfill (17). $\square$

With the help of this Lemma, we prove the main result of the section.

Theorem 3. *Let a regular polygon with vertices $x_r + iy_r = e^{ir\frac{2\pi}{m}}$ and a corresponding piecewise linear parameterization $f^{(s-1)}$ with equidistant knots $t_r = r\frac{2\pi}{m}$ be given (see Section 2).*

Let f be a periodic spline with piecewise polynomials of degree s such that $f^{(s-1)}$ is the $s-1$-th derivative of f. Such a function f can be obtained by iterative computation of antiderivatives of $f^{(s-1)}$ as previously described: Since $\int_0^{2\pi} f^{(s-1)}(t)\,dt = 0$, function $f^{(s-2)}(t) := c + \int_0^t f^{(s-1)}(u)\,du$ is 2π-periodic. Then let $f^{(s-3)}(t) := c - t\frac{\int_0^{2\pi} f^{(s-2)}(u)\,du}{2\pi} + \int_0^t f^{(s-2)}(u)\,du$, etc.

For these counter examples, estimate (6) is best possible in the sense of

$$|R(t,M)| \neq o\left(\frac{1}{M^{s+1}}\right) \text{ for all } t \in [0,2\pi) \tag{18}$$

i.e. for each $t \in [0,2\pi)$ there holds true

$$\limsup_{M\to\infty} M^{s+1} |R(t,M)| > 0.$$

Due to properties of Fourier coefficients, scaling, rotation and translation of a polygon (case $s = 1$) does not change the result. Thus sharpness is established especially for parameterizations of all regular polygons.

Proof. The outline of the proof is as follows: We first simplify formulas for arbitrary splines with respect to the counter examples of the theorem. For odd s, sharpness at knots t_r follows from (2). For even s we give an explicit estimate (21) for the knots as well as an explicit bound (23) for midpoints between knots. Then, with $D_s := \{\frac{2\pi}{m}\frac{p}{q} : \frac{p}{q} \in \mathbb{Q}, 0 < \frac{p}{q} < 1\}$, we prove sharpness for each small δ, $0 < \delta < \frac{\pi}{2m}$ on a set $D_s := D\cap[\delta, \frac{2\pi}{m}-\delta]$ if s is odd and on intervals $D_s := D\cap[\delta, \frac{\pi}{m}-\delta]$ and $D\cap[\frac{\pi}{m}+\delta, \frac{2\pi}{m}-\delta]$ if s is even. Without restriction, we only investigate D_s in the even case because the proof is not different for $D\cap[\frac{\pi}{m}+\delta, \frac{2\pi}{m}-\delta]$. The next step is to apply Lemma 2 to extend the sharpness result to the intervals $[\delta, \frac{2\pi}{m}-\delta]$ and $[\delta, \frac{\pi}{m}-\delta]\cup[\frac{\pi}{m}+\delta, \frac{2\pi}{m}-\delta]$, respectively. Since δ can be chosen arbitrarily, sharpness follows for all t in $\left(0, \frac{2\pi}{m}\right)$ or $\left(0, \frac{\pi}{m}\right)\cup\left(\frac{\pi}{m}, \frac{2\pi}{m}\right)$. In the same manner, sharpness can be shown for all sets $\left(r\frac{2\pi}{m}, (r+1)\frac{2\pi}{m}\right)$ or $\left(r\frac{2\pi}{m}, \left(r+\frac{1}{2}\right)\frac{2\pi}{m}\right)\cup\left(\left(r+\frac{1}{2}\right)\frac{2\pi}{m}, (r+1)\frac{2\pi}{m}\right)$, $0\le r<m$. Together with estimates for knots and midpoints, this is (18).

Fourier coefficients $\frac{c_k}{(ik)^{s-1}}$, $k\neq 0$, of f can be simplified, see (8), (5):

$$\begin{aligned} c_k &= \frac{1}{2\pi}\frac{1}{k^2}\sum_{r=0}^{m-1}\frac{m}{2\pi}\left[-e^{i(r-1)\frac{2\pi}{m}}+2e^{ir\frac{2\pi}{m}}-e^{i(r+1)\frac{2\pi}{m}}\right]e^{-ikr\frac{2\pi}{m}} \\ &= \frac{m}{(2\pi)^2}\frac{1}{k^2}\left[2-e^{-i\frac{2\pi}{m}}-e^{i\frac{2\pi}{m}}\right]\sum_{r=0}^{m-1}e^{-i(k-1)r\frac{2\pi}{m}}. \end{aligned}$$

If $k-1$ is no multiple of m, then $e^{-i(k-1)\frac{2\pi}{m}}\neq 1$, and

$$\sum_{r=0}^{m-1}e^{-i(k-1)r\frac{2\pi}{m}} = \frac{1-e^{-i(k-1)2\pi}}{1-e^{-i(k-1)\frac{2\pi}{m}}} = 0.$$

Thus for $j\in\mathbb{N}_0 := \{0,1,2,\dots\}$

$$c_k = \begin{cases} \frac{m^2}{(2\pi)^2}\frac{1}{k^2}\underbrace{\left[2-e^{-i\frac{2\pi}{m}}-e^{i\frac{2\pi}{m}}\right]}_{\geq 0}, & k=\pm jm+1, \\ 0, & \text{else.} \end{cases}$$

We have to consider $j\in\mathbb{N}$ that fulfill $jm+1\geq M+1 \iff j\geq\frac{M}{m}$ and $-jm+1\leq -M-1 \iff j\geq\frac{M+2}{m}$. Let $M+2$ be a multiple of $m>2$. Then each inequality is exactly fulfilled for $j\geq\frac{M+2}{m}$.

$$\begin{aligned} |R(t,M)| &= \left|\sum_{k=M+1}^{\infty}\frac{c_{-k}}{(-ik)^{s-1}}e^{-ikt}+\frac{c_k}{(ik)^{s-1}}e^{ikt}\right| \\ &= \left|\sum_{j=\frac{M+2}{m}}^{\infty}\frac{c_{-jm+1}}{(-jm+1)^{s-1}}e^{i(-jm+1)t}+\frac{c_{jm+1}}{(jm+1)^{s-1}}e^{i(jm+1)t}\right| \\ &= C\left|\sum_{j=\frac{M+2}{m}}^{\infty}\frac{e^{-ijmt}}{(-jm+1)^{s+1}}+\frac{e^{ijmt}}{(jm+1)^{s+1}}\right| \qquad (19) \\ &=: C|S|\geq C\max\{|\mathrm{Re}(S)|, |\mathrm{Im}(S)|\} \qquad (20) \end{aligned}$$

with a constant

$$C := \frac{m^2}{(2\pi)^2}\left[2 - e^{-i\frac{2\pi}{m}} - e^{i\frac{2\pi}{m}}\right]\underbrace{|e^{it}|}_{=1} = \frac{2m^2}{(2\pi)^2}\left[1-\cos\left(\frac{2\pi}{m}\right)\right] > 0.$$

At this point we deal with arguments t that are multiples of $\frac{\pi}{m}$ for even s. They have to be excluded in later considerations. For even s and knots $t_r = r\frac{2\pi}{m}$ we obtain a lower bound from (19) similar to previous estimates:

$$\begin{aligned}|R(t_r,M)| &= C\left|\sum_{j=\frac{M+2}{m}}^{\infty}\frac{-1}{(jm-1)^{s+1}} + \frac{1}{(jm+1)^{s+1}}\right| \\ &= C\left|\sum_{j=\frac{M+2}{m}}^{\infty}\frac{-2\sum_{\lambda=0}^{\frac{s}{2}}\binom{s+1}{2\lambda}(jm)^{2\lambda}}{((jm)^2-1)^{s+1}}\right| \neq o\left(\frac{1}{M^{s+1}}\right). \end{aligned} \tag{21}$$

Using partial summation (see (12)), we find for even s at the midpoints $t = r\frac{2\pi}{m} + \frac{\pi}{m}$ between knots for the values $M = lm-2$, $l \in \mathbb{N}$, that

$$|R(t,M)| = C\left|\sum_{j=\frac{M+2}{m}}^{\infty}\frac{-2e^{ij\pi}\sum_{\lambda=0}^{\frac{s}{2}}\binom{s+1}{2\lambda}(jm)^{2\lambda}}{((jm)^2-1)^{s+1}}\right| \leq \tilde{C}\frac{1}{M^{s+2}}. \tag{22}$$

But if we estimate the error for M replaced by $M+1 = lm-1$ then we loose the summand $\frac{-Ce^{i\frac{M+2}{m}\pi}}{(M+1)^{s+1}} = \frac{C(-1)^{lm+1}}{(M+1)^{s+1}}$. Because of (22) we get sharpness ad midpoints t:

$$|R(t,M+1)| \geq \frac{C}{(M+1)^{s+1}} - \tilde{C}\frac{1}{M^{s+2}}, \tag{23}$$

i.e. $|R(t,M+1)| \neq o\left(\frac{1}{(M+1)^{s+1}}\right)$.

Now we continue to estimate (20) to obtain estimates for all other points. For odd s we give a lower bound of $|\mathrm{Re}(S)|$, and for even s we estimate $|\mathrm{Im}(S)|$. Thus, for odd s we receive

$$\begin{aligned}\mathrm{Re}(S) &= \sum_{j=\frac{M+2}{m}}^{\infty}\cos(jmt)\left[\frac{1}{(-jm+1)^{s+1}} + \frac{1}{(jm+1)^{s+1}}\right] \\ &= \sum_{j=\frac{M+2}{m}}^{\infty}\cos(jmt)\frac{(jm+1)^{s+1}+(jm-1)^{s+1}}{((jm)^2-1)^{s+1}}. \end{aligned}$$

For all $t \neq t_r$ we continue:

$$\begin{aligned}
\mathrm{Re}(S) &= \mathrm{Re}\left(\sum_{j=\frac{M+2}{m}}^{\infty} e^{ijmt}\,\frac{(jm+1)^{s+1}+(jm-1)^{s+1}}{((jm)^2-1)^{s+1}}\right)\\
&= \mathrm{Re}\left(\frac{1}{1-e^{imt}}\sum_{j=\frac{M+2}{m}}^{\infty}[e^{i(M+2)t}-e^{im(j+1)t}]h(j)\right) \qquad (24)
\end{aligned}$$

due to (11) with

$$h(j) := \frac{(jm+1)^{s+1}+(jm-1)^{s+1}}{((jm)^2-1)^{s+1}} - \frac{((j+1)m+1)^{s+1}+((j+1)m-1)^{s+1}}{(((j+1)m)^2-1)^{s+1}}.$$

If one uses a common denominator, the highest power of j in the numerator is $3s+2$ with a coefficient $2(s+1)m^{3s+3}$. The highest power of the denominator is $4s+4$ with coefficient m^{4s+4}. Asymptotically, $h(j)$ behaves like $\frac{2(s+1)}{m^{s+1}}\frac{1}{j^{s+2}}$.

For even s, we also get (24) but with Re replaced by Im. Also, for odd and even s we can show with (12) that for a constant C_0, independent of t and M,

$$\frac{1}{|1-e^{imt}|}\left|\sum_{j=\frac{M+2}{m}}^{\infty} e^{im(j+1)t}h(j)\right| \le \frac{C_0}{|1-e^{imt}|^2}\frac{1}{M^{s+2}}. \qquad (25)$$

Thus it is sufficient for odd s to find a lower bound for

$$\begin{aligned}
&\left|\mathrm{Re}\left(\frac{e^{i(M+2)t}}{1-e^{imt}}\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right)\right|\\
&= \left|\mathrm{Re}\left(\frac{e^{i(M+2)t}(1-e^{-imt})}{(1-e^{imt})(1-e^{-imt})}\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right)\right| = \frac{c_{\mathrm{odd}}(M)}{2-2\cos(mt)}\left|\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right|
\end{aligned}$$

with

$$\begin{aligned}
c_{\mathrm{odd}}(M) &:= |\cos((M+2)t)(1-\cos(mt))-\sin((M+2)t)\sin(mt)|\\
&= |\cos(lmt)(1-\cos(mt))-\sin(lmt)\sin(mt)|
\end{aligned}$$

for a number $l \in \mathbb{N}$.

If s is even, we have to estimate

$$\left|\mathrm{Im}\left(\frac{e^{i(M+2)t}}{1-e^{imt}}\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right)\right| = \frac{c_{\mathrm{even}}(M)}{2-2\cos(mt)}\left|\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right|$$

with

$$c_{\mathrm{even}}(M) = |\sin(lmt)(1-\cos(mt))+\cos(lmt)\sin(mt)|.$$

Since $\left|\sum_{j=\frac{M+2}{m}}^{\infty} h(j)\right| \in \Omega\left(\frac{1}{M^{s+1}}\right)$, it remains to show $c_{\text{odd/even}}(M) \neq o(1)$, i.e. $c_{\text{odd/even}}(lm-2) \neq o(1)$, $l \to \infty$, for the points t under consideration.

Now we focus on $t \in D_s$. For each such $t = \frac{2\pi}{m}\frac{p}{q}$ we discuss an individual subsequence of $M_l = lm-2$ such that l is a multiple of q. Then for all such l:

$$\begin{aligned}
&|\cos(lmt)(1-\cos(mt)) - \sin(lmt)\sin(mt)| \\
&= \left|\cos(2\pi)\left(1-\cos\left(2\pi\frac{p}{q}\right)\right) - \sin(2\pi)\sin\left(2\pi\frac{p}{q}\right)\right| = \left(1-\cos\left(2\pi\frac{p}{q}\right)\right) \\
&> (1-\cos(m\delta)) =: c_0 > 0.
\end{aligned}$$

This also holds for $\frac{p}{q} = \frac{1}{2}$. But we have to exclude this value for even s, that is why D_s is defined differently.

$$\begin{aligned}
&|\sin(lmt)(1-\cos(mt)) + \cos(lmt)\sin(mt)| \\
&= \left|\sin(2\pi)\left(1-\cos\left(2\pi\frac{p}{q}\right)\right) + \cos(2\pi)\sin\left(2\pi\frac{p}{q}\right)\right| = \left|\sin\left(2\pi\frac{p}{q}\right)\right| \\
&> \sin(m\delta) =: c_1 > 0.
\end{aligned}$$

For $t \in D_s$, factor $\frac{C_0}{|1-e^{imt}|^2}$ in (25) is bounded by $C_1 := \frac{C_0}{|1-e^{im\delta}|^2}$. Constants C_1, c_0, and c_1 are all independent of $t \in D_s$, but subsequences of M_l do depend on t and are now denoted by $(M_{t,j})_{j=1}^{\infty}$. Thus

$$\left|R(t, M_{t,j})\right| \geq \frac{c}{M_{t,j}^{s+1}} \text{ for each } t \in D_s$$

with a constant c that is dependent on δ but independent of t. Therefore, we can apply Lemma 2 with $[a,b] := [\delta, \frac{2\pi}{m} - \delta]$ or $[a,b] := [\delta, \frac{\pi}{m} - \delta]$, $[a,b] := [\frac{\pi}{m} + \delta, \frac{2\pi}{m} - \delta]$ and $R_M(t) := \frac{|R(t,M)|}{M^{s+1}}$ to obtain the result. □

Since we only work with a subsequence of values M, we cannot follow that the error is in $\Omega\left(\frac{1}{M^{s+1}}\right)$. Such a lower bound cannot be expected, because the low pass filtered curve intersects with the spline, see Figure 1. The error might become zero at certain points for the given M.

It is not possible to extend Theorem 3 from regular to arbitrary polygons. For example, the Fourier partial sums of an odd, real-valued, periodic function only consist of sine wave summands. At $t = 0$ both function and partial sums are zero, the error vanishes, see Figure 3.

5 Conclusion and Application

We have given bounds for the approximation of periodic splines by Fourier partial sums and have shown their sharpness. However, Theorem 3 covers parameteriza-

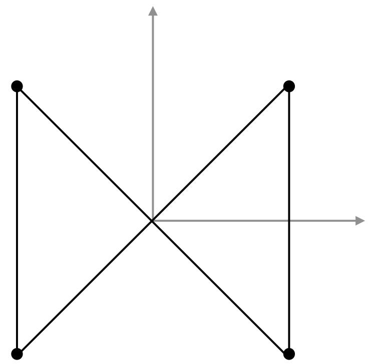

Fig. 3 Starting at the origin, the polygon can be parameterized by a function f with odd component functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. Similar to the counter examples of Theorem 3, equidistant knots can be used at which the parameterization is not differentiable. Also, the parameterization can be translated such that $t_0 = 0$ is a knot. Then, at the translated origin, the approximation error of all Fourier partial sums is zero.

tions based on regular polygons only. It remains open under which more general assumptions sharpness $|R(t,M)| \neq o\left(\frac{1}{M^{s+1}}\right)$ at all points $t \neq t_r$ holds true.

Since the functions are not arbitrarily often differentiable at knots, their spectrum is not bounded. Especially the approximation at knots requires high frequencies. Thus it is somewhat surprising that the rate of convergence at knots for even degree splines is not worse than at other points. On the other hand, the slower convergence

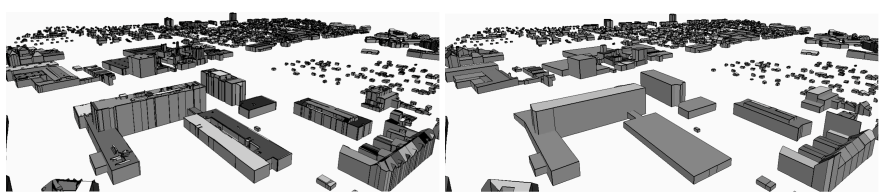

Fig. 4 Footprint polygons are simplified according to the Fourier partial sum approximation error in knots. Iteratively, the vertex with best approximation is removed until the polygon's area falls below a threshold value. In addition to merging roof facets, this method is used to reduce the level of detail in the 3D city model.

rate at knots for odd degree splines f can be utilized to define dominant corners of contours that are represented as polygons or splines. According to (3), the size of the error at a knot t_r is determined by $|\alpha_r|$. In case of an equidistant parameterization $f^{(s-1)}$ of a polygon, the second difference in (5) can be interpreted as the sum

$$(x_{r-1}, y_{r-1}) - (x_r, y_r) + (x_{r+1}, y_{r+1}) - (x_r, y_r)$$

of the two edge vectors, scaled by $-\frac{m}{2\pi}$ (cf. Section 2). Let $l_1 := |(x_{r-1}, y_{r-1}) - (x_r, y_r)|$, $l_2 := |(x_{r+1}, y_{r+1}) - (x_r, y_r)|$, and $0 \leq \gamma \leq \pi$ the angle between the two edges. Then the cosine theorem allows understanding the role of α_r:

$$\frac{4\pi^2}{m^2}|\alpha_r|^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\pi - \gamma).$$

Asymptotically, the error becomes large for large edges and small angles γ. Such vertices appear to be rather dominant and more important for recognition of contours than others. If one uses a constant speed parameterization (see Section 2), then vectors become normed, and $|\alpha_r|$ only depends on the angle: $|\alpha_r|^2 = 2 - 2\cos(\pi - \gamma)$.

Literally hundreds of methods for dominant corner detection exist either based on grey values of images or extracted contours. For contour-based methods, [1] and [9, Sections 3 and 4] give an overview. Common is a smoothing step that removes noise. When using the approximation error at knots as a measure for dominance, we obtain "better" results than when directly using $|\alpha_r|$ as measure: The right picture in Figure 1 shows larger errors at the four edges of the underlying square although all $|\alpha_r|$ are equal. Fourier low-pass filtering removes noise. Higher order terms in (3) contribute to this effect. Fourier partial sums represent global approximations of curves' shapes. In contrast to this, lengths and angles of edges are local features. A large distant to the global approximation of a shape especially indicates a dominant edge. A related local corner detector is described in [12]. It applies a wavelet decomposition to the boundary curve and looks for large wavelet coefficients that describe local changes. By omitting corresponding terms of the inverse wavelet transform, the approximation error will also become large.

An application of our partial-sum-based dominant corner detector is shown in Figure 4: All non-dominant corners were removed from a 3D city model to reduce the level of detail.

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